Lecture 20 - 4-color Theorem

Thursday, April 9, 2026 8:27 AM

HW6 p4

↳ Show all planar graphs are the subgraph of a triangulation

p5 → maximal planar graphs ↔ triangulations

Why we care:

- Kuratowski
- 4/5-color theorems

↳ these centrally use triangulations

If we prove for triangulations → we prove for all (planar) graphs

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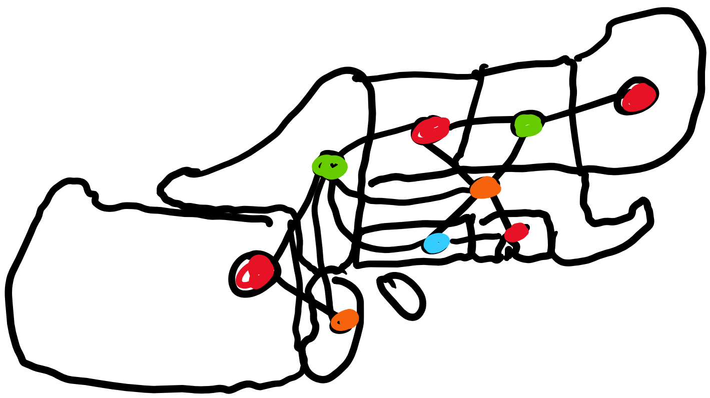
Motivation: map coloring



↓ Q: How many colors to properly color the faces of a map?

→ Equivalent to coloring a planar graph

Q: How many colors are needed for a planar graph?

sub Q: can we bound $\chi(G)$ for some planar G?

5-color theorem: yes we can

↳ $\chi(G) \leq 5$

Note: any counter-example

any counter-example

must have some v s.t. $d(v) \le 5$

$\hookrightarrow$ recall: $m \le 6n - 6$

and note: $\sum_{v}^{n} d(v) = 2m$

if all $d(v) = 6$

$\rightarrow 2m = 6n$

$m \le 3n - 6$

$2m \le 6n - 12$

$6\cancel{n} \le \cancel{6}n - 12$

$0 \le -12$ ✗

$\Rightarrow \exists v : d(v) \le 5$ in all planar graphs (or triangulations)

<u>5-color theorem</u>: $\chi(G) \leq 5$

if G is a triangulation

To show: strong induction on $|V(G)|$

Basis: $P(\leq 5) \rightarrow$ trivial to 5-color

$P(n)$ is a triangulation with n vertices

$P(k) = P(n) - v$

where $d(v) \leq 5$

From Kuratowski proof

$\hookrightarrow$ vertex deletion won't create a K.S. $\rightarrow$ $P(k)$ is planar

(can be triangulation)

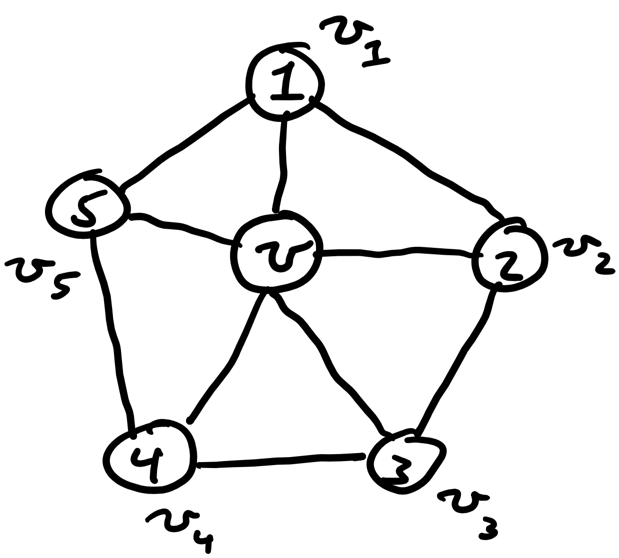
I.H. on $P(k) \rightarrow$ 5-coloring

Bring it on back to $P(n)$

Case 1: $d(v) \leq 4 \rightarrow$ trivial via G.C.

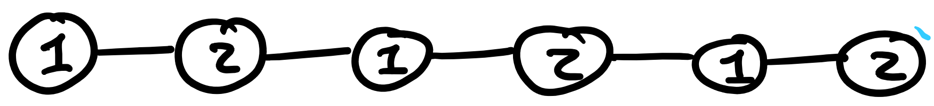
Case 2: $d(v) = 5$ and fewer than 5 colors in $N(v) \rightarrow$ trivial

Case 3: $d(v)$ and 5 colors in $N(v)$

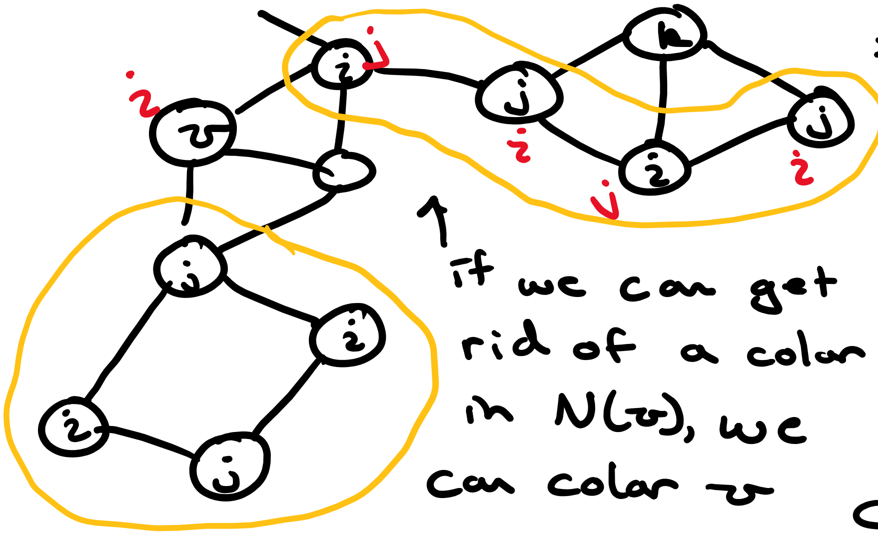


Show: this non-avoidable configuration can be reduced with only 5 colors

To do so: Kempe Chains aka color-alternating paths



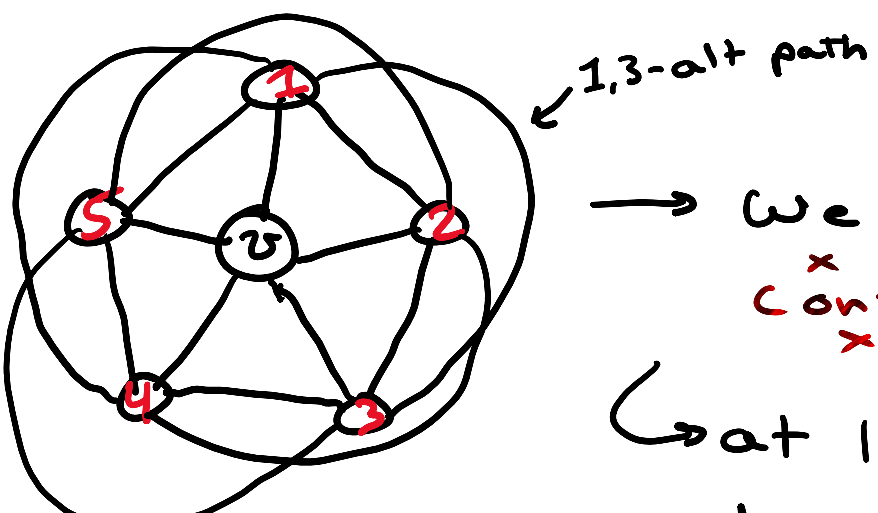
Consider all possible Kempe chains around v for all possible i,j color pairs



↑ if we can get rid of a color in $N(v)$, we can color v properly with that color

* if for a given pair of colors, no i,j-alternating path exists, we can swap the colors on a maximal i,j-induced subgraph

Q: will there be an i,j-alternating path for all possible i,j?



→ we have a K.S.

contradiction

↳ at least one i,j does not have connecting Kempe chain

$P(k) \to P(n)$: we select some i,j pair
w/o a Kempe chain,
swap colors on one
maximal i,j subgraphs
$\to$ color v with color now
not in $N(v)$ $\square$

What about 4-colors

Gary: Let's give it a go
with the same approach

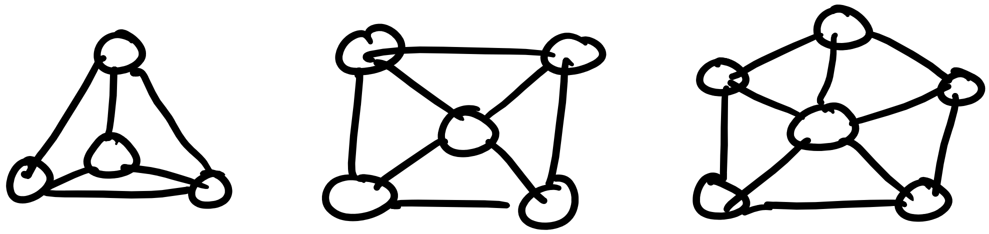
$\to$ We're looking to find some
unavoidable configuration that
a counter-example must contain

If we show all configurations are
reducible to color v w/ $1 \ldots 4$

$\Rightarrow$ all planar G are 4-colorable

Our 5-color theorem configurations

Our 5-color theorem configurations



Let's try the same approach

Basis $P(\leq 4) \rightarrow$ trivial

$P(n)$: planar G with $v : d(v) \leq 5$
(triangulation)

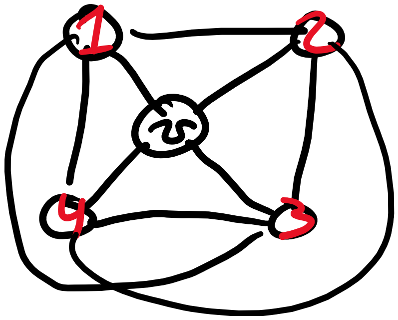
$P(k) = P(n) - v$

I.H.: $P(k)$ is 4-colorable

Bring it back to $P(n)$

Case 1: $d(v) = 3 \rightarrow$ trivial

Case 2: $d(v) = 4$



$\rightarrow$ at least one i, j pair must have no connecting i, j Kempe ch...

connecting i, j Kempe chain

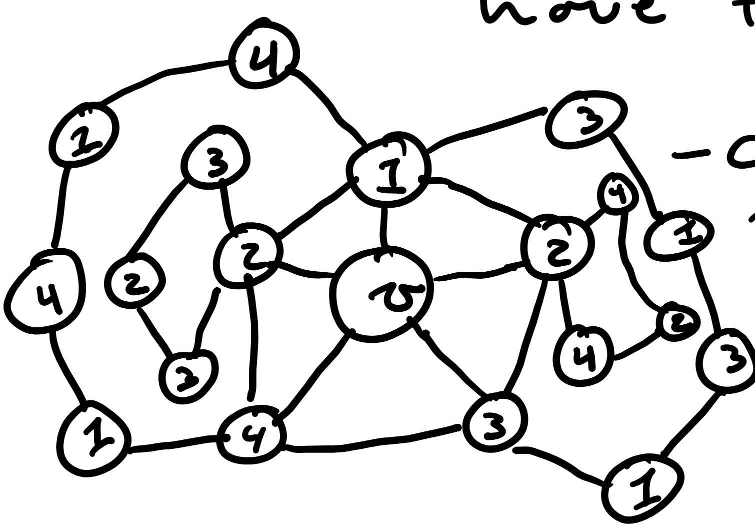
(otherwise → K_5 K.S.)

→ we can reduce and give v color i (or j)

Case 3: $d(v) = 5$ and fewer than 4 colors → trivial

Case 4: $d(v) = 5$ and 4 colors in $N(v)$

Note: 2 vertices in $N(v)$ will have the same color



– consider paths from $1 \to 3$ and $1 \to 4$

we can eliminate color 1 if these paths don't exist

Now consider vertices w/ color 2

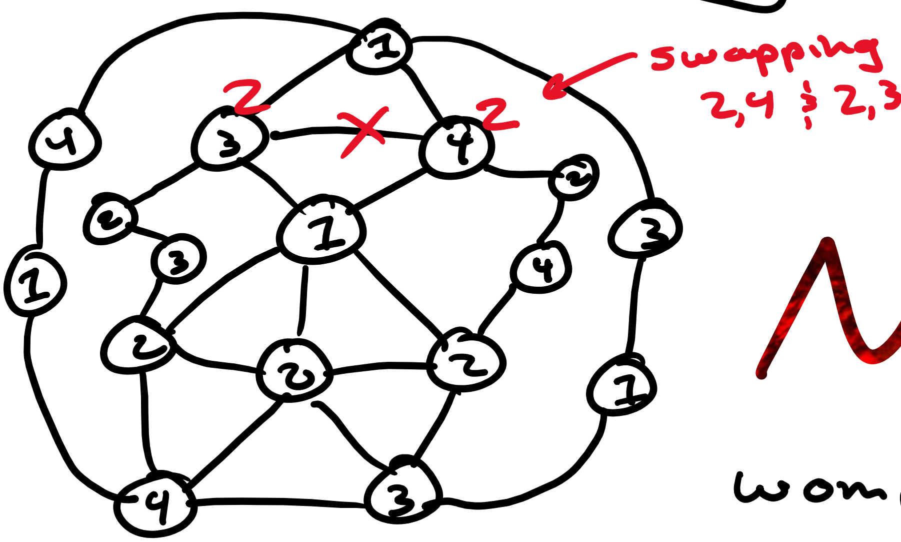
→ As we can't have a 2,3-alternating

→ As we can't have a 2,3-alternating path, we can swap colors 2,3 to get rid of one color 2 vert in N(v)

→ Do the same for 2,4

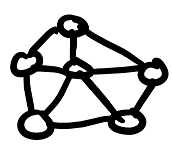
→ Give vertex v color 2 ☑ QED

Actually no



NOPE

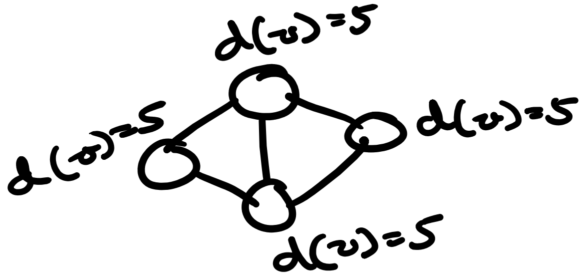
womp womp womp

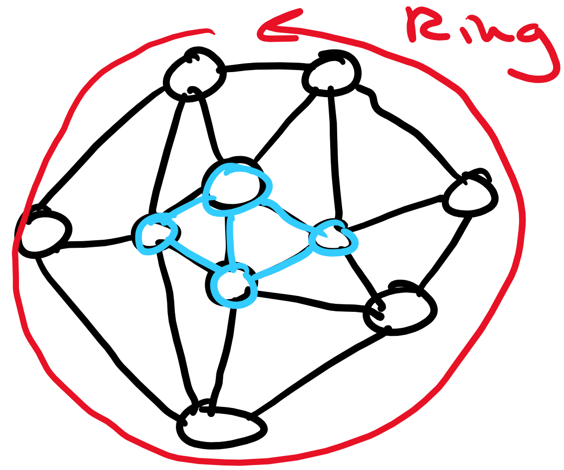
↳ This configuration cannot be reduced 

→ We need to consider more than a single vertex for

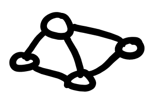
that a single vertex (a
our configurations

E.g. Birkhoff Diamond

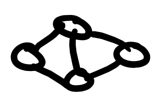




↳ we need to reduce this configuration using the same logic

$P(k) = P(n) -$ 

I.H. on $P(k)$

$P(n) \to P(k)$: show a proper 4-coloring can be extended back to 
(using Kempe-chain or the like)

The actual 4-color theorem:

- Generate all necessory configurations
- Show all of them are reducible

Q: How many configurations:

O.G. proof: ~ 1800 (womp womp)

Now: ~ 600 (womp)

Q2: How can we generate our configurations?

$$\sum_{v}^{n} d(v) = 6n - 12$$

$$\hookrightarrow 12 = \sum_{v}^{n} (6 - d(v))$$

"charge" of v

As +12 sum is fixed

↳ at least some vertices have a positive charge

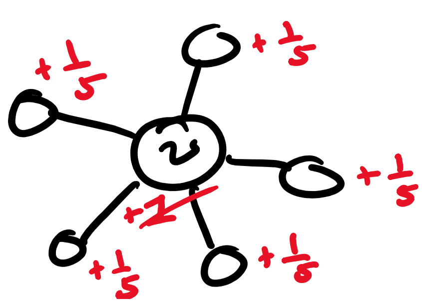
We can generate configurations by considering how charge is distributed

↳ <u>discharge rules</u>

Consider some charge in a triangulation with $\delta(G) \geq 5$

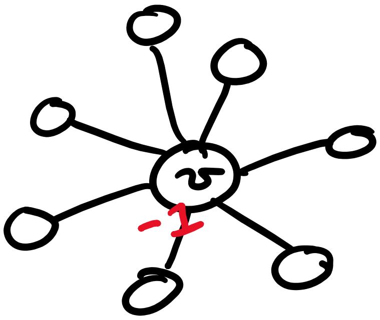
Our first discharge rule:

→ take and distribute some positive charge on v equally to all of $N(v)$



← if one of these $N(v)$ has a positive charge, it must have a degree

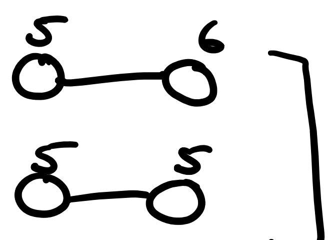
has a positive charge, it must have a degree of 5 or 6



→ if any vertex ends up with a positive charge, there must be at least 6 $d(u)=5$ vertices in the neighborhood

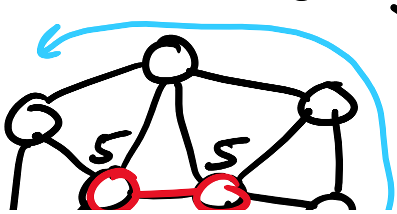
Note: in $N(v)$, at least 2 of the degree-5 vertices must be connected (same with degree-6)

=> this gives us

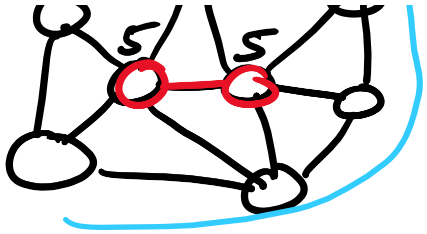


for induction

→ we also need to consider varying ring sizes

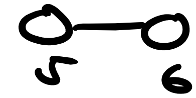


→ we reduce this



ring of size 6

→ we reduce this configuration

→ repeat for 

→ consider other discharge possibilities

→ reduce those too

⇒ Now we're done $\square$

WP10: use result that planar G have $\chi(G) \leq 4$

① Construction that is still planar?

② Note: can't just take e.g. colors 1,3 and 2,4 and induce subgraphs

→ decompositions require

decompositions require
all edges in H_1 or H_2
(exclusively)

Q: How to assign edges
to H_1 or H_2?

(edge coloring?)
↳ using endpoints

Can we guarantee only
even cycles?

(2 alternating vertex
classes on each edge)