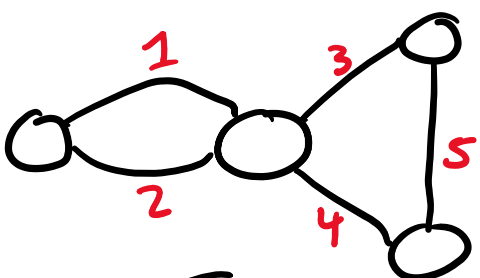
Slota's biggest regret: not becoming a professional ultimate frisbee player.

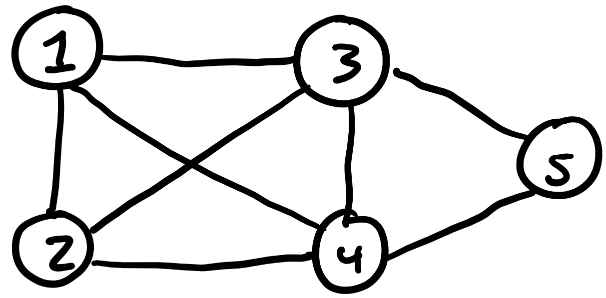
# Line graphs

the line graph of $G \to L(G)$

defined as $\begin{cases} E(G) \to V(L(G)) \\ E(L(G)) \text{ are defined via shared endpoints of edges in } G \end{cases}$




G




$L(G)$

Note: each vertex $v$ in $G$ becomes clique $K_{d(v)}$ in $L(G)$

Note 2: the equivalence of

$E(G) \to V(L(G))$ ...

Note 2: the equivalence of $E(G) \to V(L(G))$ is relevant to several problems

1. Euler Tour on G (EZ)

$\iff$

Spanning cycle on $L(G)$ (hard)

2. Matching on G (EZ)

$\iff$

independent set on $L(G)$ (hard)

3. Cut edge on G EZ

$\iff$

cut vertex on $L(G)$ EZ

4. edge coloring on G (EZ?)

$\iff$

vertex coloring on $L(G)$
(hard)

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## Edge Coloring

Assign labels to all $e \in E(G)$

<u>proper</u>: no two edges that share an endpoint have the same label (i.e., color)

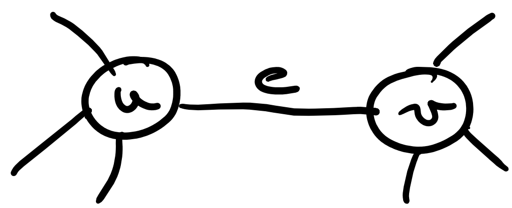
Edge Chromatic Number $\chi'(G)$

→ minimum number of colors to properly edge-color $G$

Let's Get Poundin'

$\chi'(G) \geq \Delta(G)$ as the largest degree vertex require unique colors for each incident edge

Consider Greedy Coloring




$$\chi'(G) \leq 2\Delta(G) - 1$$

Note: k-regular graphs have a perfect match

↳ all bipartite graphs are the subgraph of some k-regular bipartite graph

↳ Color our P.M., remove it, color a new P.M.,

", color a new P.M.,
remove and repeat $k$
times

$\Rightarrow$ we have a $k$-edge-coloring
on $G$

$$\chi'(G) = \Delta(G)$$

For bipartite graphs

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Q: Can we tighten our lower
bound in general?

A: yup

$\Rightarrow$ show $\chi'(G) = \Delta(G)$

or $\chi'(G) = \Delta(G) + 1$

(for simple graphs)

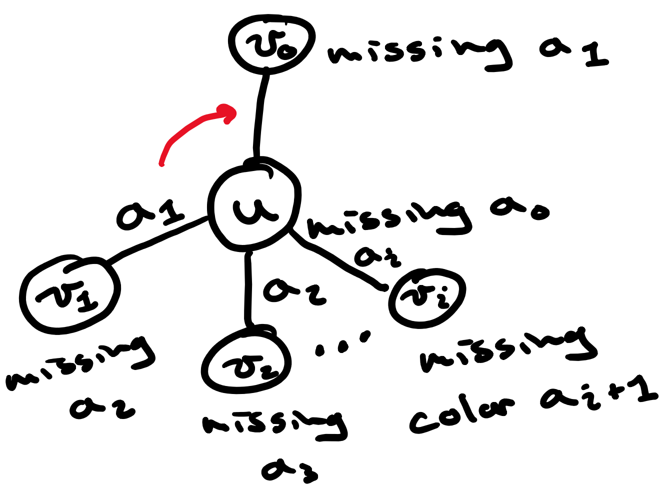
PROOF BY ALGO.

# PROOF BY ALGO.

Consider some $\Delta(G)+1$ edge-coloring of some subgraph $H \subseteq G$ $\rightarrow$ extend to all of $G$

Consider $u \in V(G)$ and edge $(u, v_0) \in E(G)$ with no color

In $N(u)$, there are some colors missing $\rightarrow$ $a_0$ is one color




- Consider $u$'s neighbors
- Label $N(u)$ s.t. $a_{i+1}$ $\rightarrow$ color missing at vertex $v_i \in N(u)$

If color $a_0$ is not $N(v_0)$

$\rightarrow$ color $(u, v_0)$ with $a_0$

If color $a_1$ is not in $N(v_0)$ and $a_1$ is not in $N(u)$

and $a_1$ is not in $N(u)$

→ color $(u, v_0)$ with $a_1$

If color $a_2$ is missing at $v_1$, there must exist $(u, v_2)$ with color $a_2$,

→ otherwise color $(u, v_0)$ w/ $a_1$

$(u, v_1)$ w/ $a_2$

Generally: if $a_i$ is missing, we can use $a_i$ on $(u, v_{i-2})$ and "shift" our colors down to eventually color $(u, v_0)$ with $a_1$

→ Either a missing color repeats or this procedure is possible since we have at most $\Delta(G)+1$ colors

$v_\ell$ is the first vertex with a missing color on $a_1 \dots a_\ell$

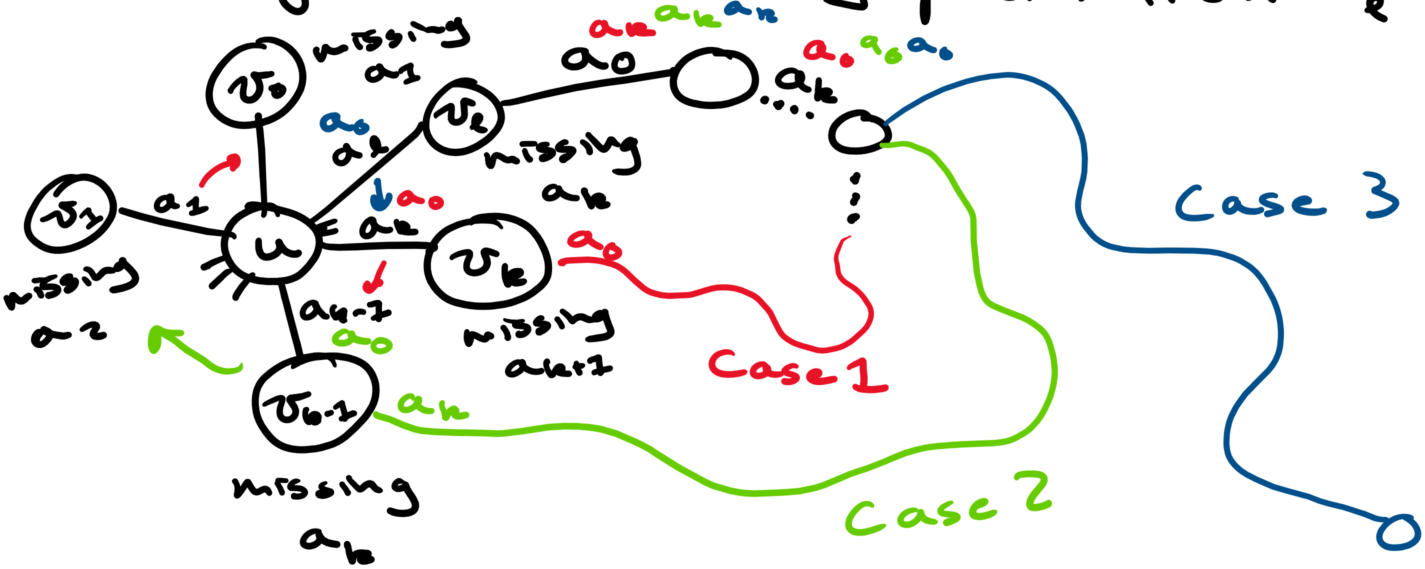
→ we'll call it $a_k$

Note: also missing at $v_{k-1}$ and it is on edge $(v_k, u)$

Note 2: $a_0$ also appears on $v_e$, otherwise we could color $(u, v_e)$ with $a_0$ and shift colors down

# Exxtremal Arg

Consider $P$ as a maximum $a_0 a_k$ edge alternating path from $v_e$




Case 1: $P$ reaches $v_k$

$\rightarrow$ Shift our colors down from $v_{k-1}$ and swap colors on $P$, edge $(u, v_e)$ getting color $a_k$

Case 2: $P$ reaches $v_{k-1}$

$\rightarrow$ Shift down from $v_{k-1}$, put $a_0$ on $(u, v_{k-1})$ and swap colors on $P$

Case 3: $P$ reaches elsewhere

$\rightarrow$ shift colors down from $v_e$, put $a_0$ on $(u, v_e)$ and swap colors on $P$

$\Rightarrow$ No matter what, all graphs can be (edge) colored in at most $\Delta(G)+1$ colors $\square$

$$\boxed{\Delta(G) \le \chi'(G) \le \Delta(G)+1}$$

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As we discussed

if $\exists H$ s.t. $L(H) = G$

→ we can solve max ind. set on $G$ in polynomial time

→ we can get an optimal vertex coloring on $G$ in poly. time

→ we can get a hammy cycle (spanning) in linear time

Our Q: given some $G$, does there exist some $H$ where $G$ is the line graph of $H$?

A: sometimes

O: if such an H existed for all G → P = NP
obviously

Q2: Can we characterize G where such an H exists?

$L(H) = G$

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From earlier

→ we observed each edge in the original graph becomes part of (up to) two cliques in the line graph

→ a line graph can decompose into maximal cliques with each vertex in at most 2

This gives us a <u>necessary</u> condition for our G

$\longrightarrow$ is it sufficient?

For simple $G$, $\exists H$ s.t. $L(H) = G$
iff $G$ decomposes into maximal cliques where each vertex is in at most 2

($\Rightarrow$) Note: every vertex in $H$ becomes a clique in $G$
And: every edge in $H$ is attached to at most 2 maximal cliques as a vertex in $G$

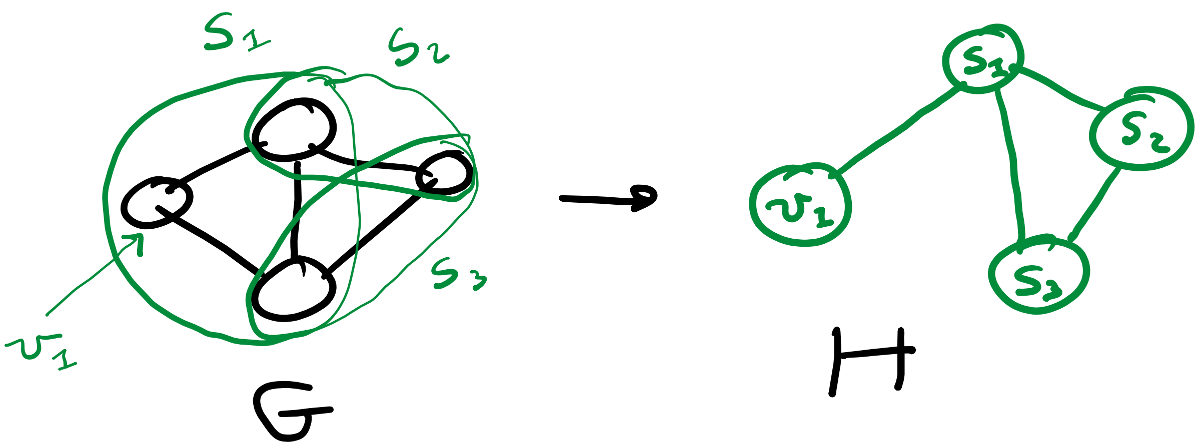
($\Leftarrow$) Construct $H$ given $G$'s decomp.

define: $S_1 S_2 \ldots S_k$ as vertex sets of maximal cliques

$v_1 v_2 \ldots v_\ell$ are vertices in only one of $S_i$

$V(H) = \{$ one vertex for each in $S_1 \ldots S_k \; v_1 \ldots v_\ell \}$

$E(H) = \{$ for all $(v_i, S_j)$ and $(S_n, S_m)$ where these pairwise intersect$\}$




$\rightarrow$ each $v \in V(H)$ is in at most two sets $S_i S_j$ with no vertices in the same two sets

$\Rightarrow$ this implies the existence of our $H$ s.t. $L(H) = G$ $\square$

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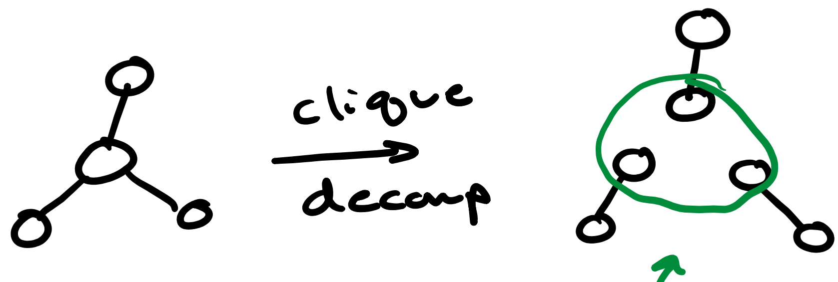
Big Q:

Big Q:

Is there a simpler characterization?

Big A: yes

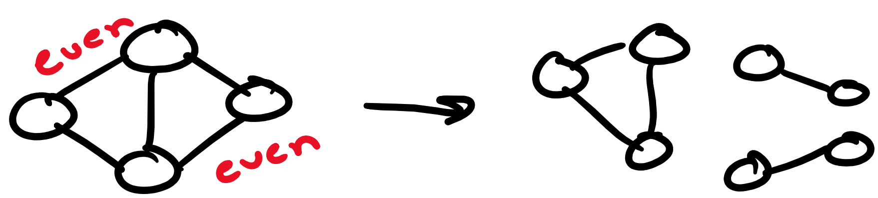
Consider an induced claw




Center $v$ is in at least 3 max cliques

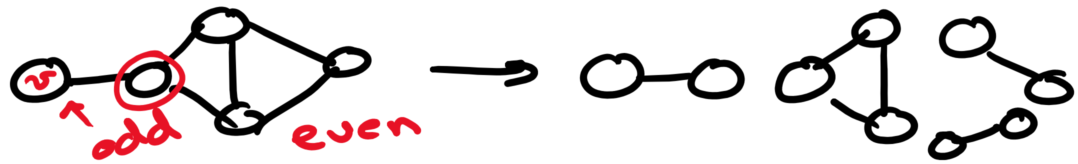
$\Rightarrow$ no such G can have an induced claw as a subgraph

Also consider a double triangle



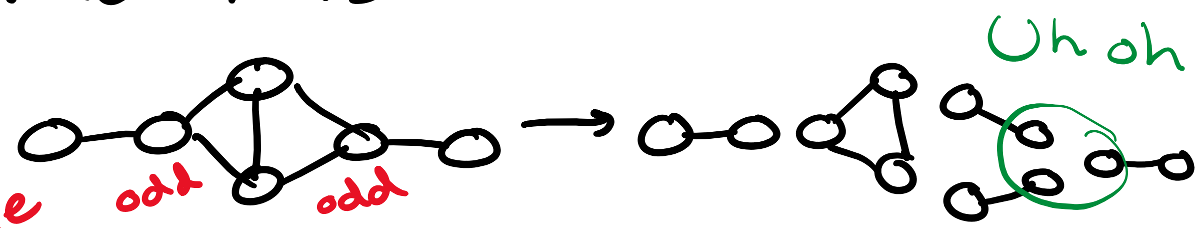

what about:

what about:




And this:

Double odd triangle




(consider as induced subgraph)

odd triangle $T$: $\exists v \in V(G)$, $T \subseteq G$ s.t. $|N(v) \cap V(T)| = \text{odd}$

even triangle $T$: $\forall v \in V(G)$, $T \subseteq G$ s.t. $|N(v) \cap V(T)| = \text{even}$

$\Rightarrow$ If $G$ has a double odd triangle, it can't properly decompose $\rightarrow$ no $H$ exists

Necessary:

$G$ has no cl

G has no claws

G has no double odd triangle

Q: sufficient?

(Next class)