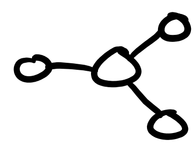
Lecture 22 - Hamiltonian Graphs

Thursday, April 16, 2026 8:43 AM

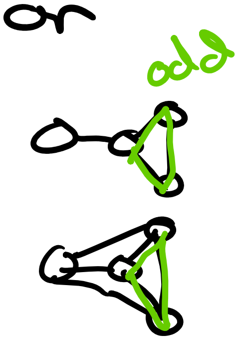
SLPT: Soft skills are just as important, if not more so, than technical skills after you're only a few years into your career.
Addendum: Try not to make a horrible impression when meeting someone the first time. Burning bridges will only serve to get you stranded on an island in a river in Konigsberg (metaphorically).

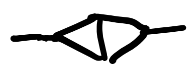
Prove: $\exists H$ s.t. $G = L(H)$

iff



G has no claws or double odd triangles





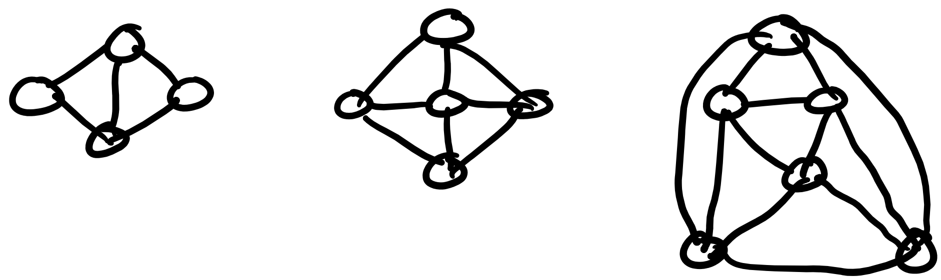
($\Rightarrow$) Contrapositive

$\Rightarrow$ we proved this last class

($\Leftarrow$) First: consider double even triangles (that are claw free)

$\rightarrow$ only 3 such graphs exist

→ only 3 such graphs exist (simple connected)



⇒ we only need to consider double triangles with one even and one odd triangle

Consider maximal clique decomp:

$S_1, S_2 \ldots S_k$ are maximal cliques except for even triangle edges that get labeled $T_1, T_2 \ldots T_\ell$

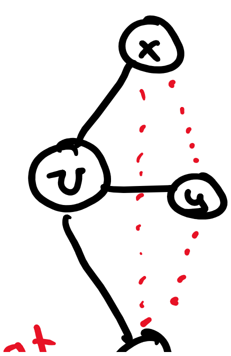
Q: $\forall v \in V(G)$: is v in at most 2?

(A, B, C in S_i or T_j)

Consider $v \in A, B, C$

$x, y, z \in N(v)$

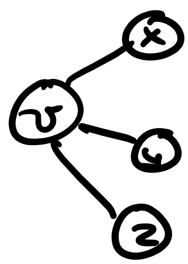
$x \in A, y \in B, z \in C$



Show: each edge cannot

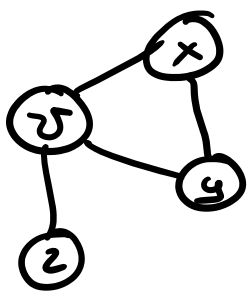
Show: each edge cannot be in unique S_i, T_j

Case 1: no edges (x,y) (x,z) (y,z)



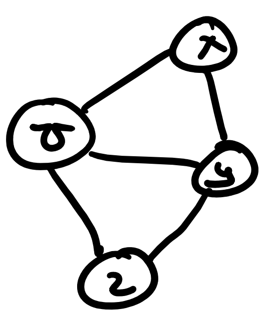
→ this is a claw, so it can't happen ✓

Case 2: wlog (x,y) exists



→ an odd triangle, which will be in same S_i

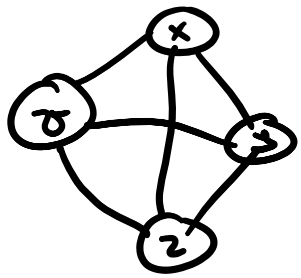
Case 3: (wlog) edges (x,y) and (y,z) exist



→ we have 2 even triangles, which we've ruled out as a possibility (at least one must be odd and in same S_i)

Case 4: all ...

Case 4: all edges exist



→ we have K_4, which will all be in same S_i

⇒ together, along with our assumed decomposition, v is in at most two of S_i T_j

⇒ $\exists H$ s.t. $G = L(H)$, using result from last class $\square$

All together:

Our characterization of G where $G = L(H)$

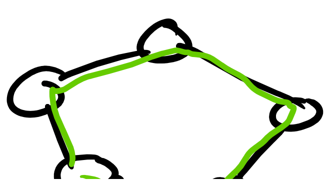
G has no claws
G has no OOTs

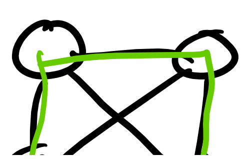
Forbidden subgraphs

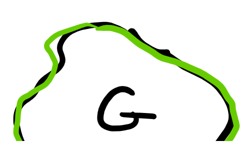
Q: What are Hamiltonian graphs?

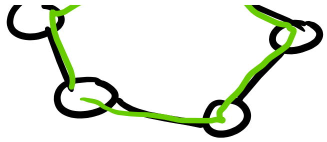
A: Graphs with a spanning cycle aka a Hamiltonian cycle

What graphs are hamiltonian?

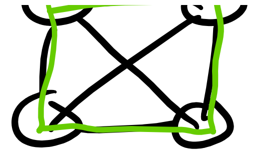




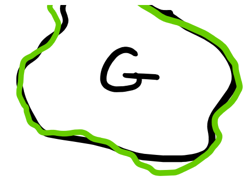




cycle graph

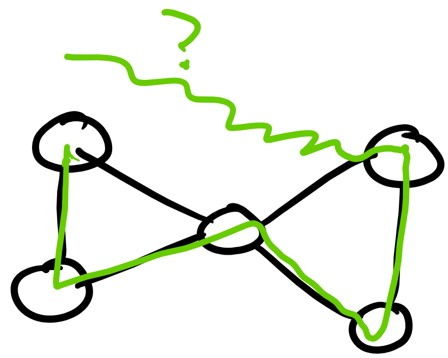


cliques

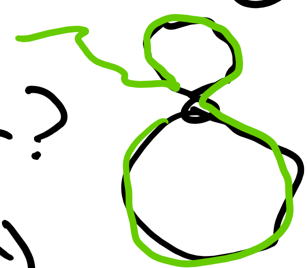


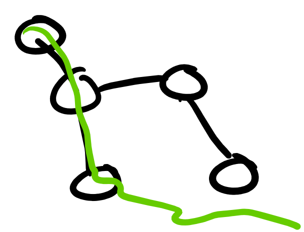
2-connected outerplanar graph

Eulerian?



Regular outerplanar?
(not 2-connected)





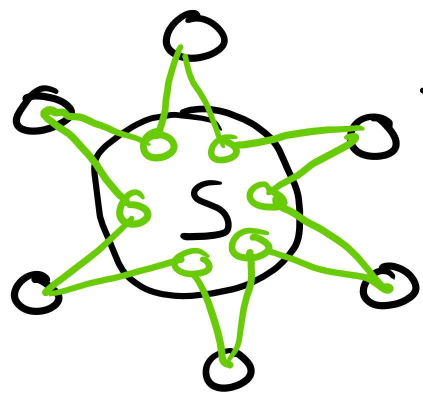
trees

Necessary conditions for Hamiltonianness (sp?) of G

- G must be connected
- G must be 2-connected
- If G is bipartite then $|X| = |Y|$

- If $c(G)$ = # of components of G

then $c(G-S) \leq |S| \; \forall S \subseteq V(G)$



→ for each comp, we need at least one $v \in S$ to construct a spanning cycle

Q: But what about sufficient conditions?

Sufficient conditions for a hammy G

If $|V(G)| \geq 3$ and $\delta(G) \geq \lceil \frac{|V(G)|}{2} \rceil$ then G is hammy

Consider maximal non-Ham. G'

Consider maximal non-Ham. G'

$\rightarrow$ $G' + e$ = Hammy cycle

$\rightarrow$ G' = Hammy path

↳ i.e., spanning path

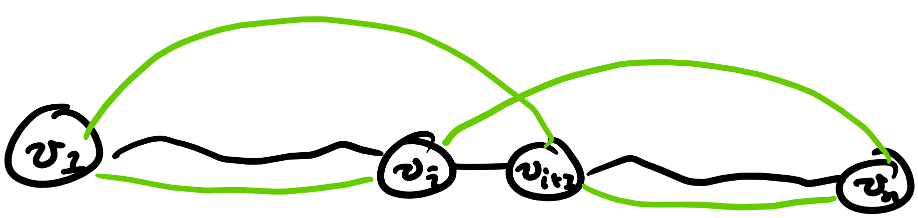
Consider this path is some order

$$P = \{v_1 v_2 \dots v_n\}$$

If along this path $\exists v_i v_{i+1}$

s.t. $v_i \in N(v_n)$ and $v_{i+1} \in N(v_1)$

$\longrightarrow$ we can construct a spanning cycle



define $S = \{i : (v_1 v_i) \in E(G)\}$

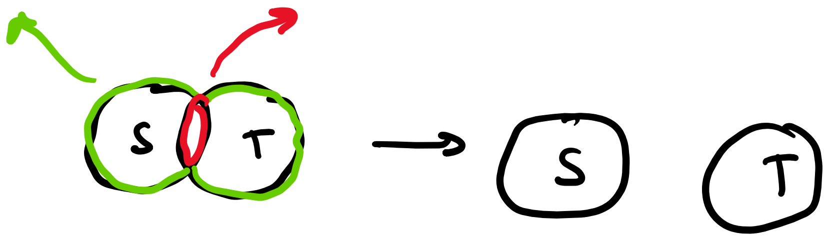
define $T = \{i : (v_n v_i) \in E(G)\}$

show $|S \cap T| \geq 1$

↳ we have a pair above

↳ we have a pair above that creates a spanning cycle

$$|S \cup T| + |S \cap T| = |S| + |T|$$



$$|S| + |T| = d(v_1) + d(v_n) \geq |V(G)|$$

(using assumed $\delta(G)$)

$$|S \cup T| + |S \cap T| \geq |V(G)|$$

$|S \cup T| < |V(G)|$ if we assume no $(v_1 v_n)$

$$|S \cap T| \geq 1$$

→ we have a spanning cycle

⇒ Consider applying this logic to all possible vertex pairs would allow the construction of would allow the construction of a spanning cycle in the general case □

If $\forall u, v \in V(G)$ $(u,v) \in E(G)$

$$d(u) + d(v) \geq |V(G)|$$

G is Hamiltonian

iff

$G + (u,v)$ is Hamiltonian

($\Rightarrow$) trivially, adding an edge won't delete a spanning cycle

($\Leftarrow$) This follows directly from our prior proof

The above can be used to define the <u>closure</u> of G

Closure of G: $\forall u, v \in V(G)$, $(u,v) \notin E(G)$
s.t. $d(u) + d(v) \geq |V(G)|$
add (u,v)

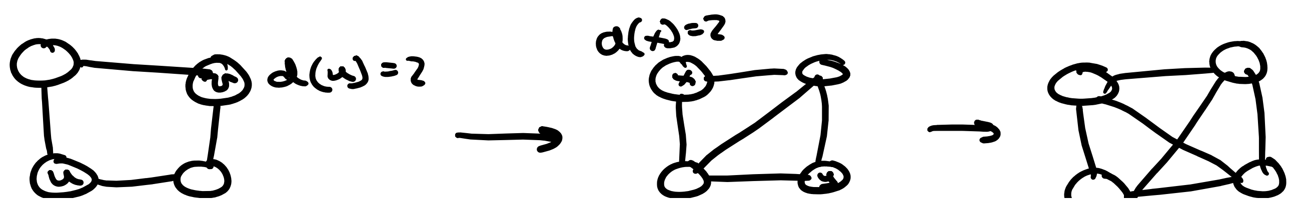
iterate until no
such u, v exist

Note: $d(u)$, $d(v)$ can increase
due to added edges

To rephrase the prior proof:

G is Hamiltonian
iff
G's closure is Hamiltonian

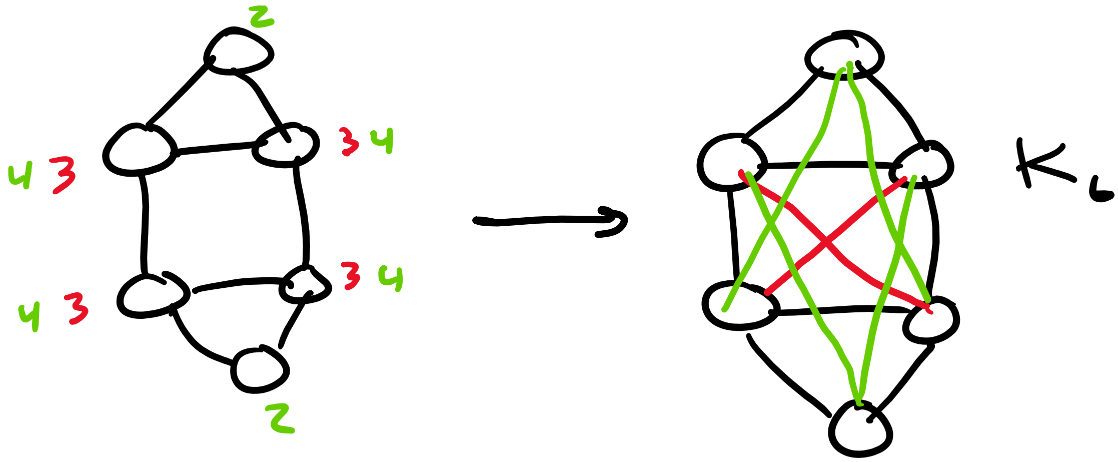
Let's LOOK at
some closures



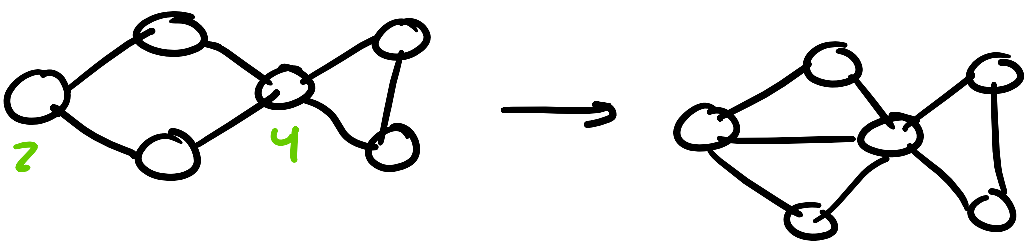
$d(u)=2$ $\quad 2+2 \geq |V(G)| = 4$ $\quad d(y)=2$

$K_4 \rightarrow$ cliques are Hamy

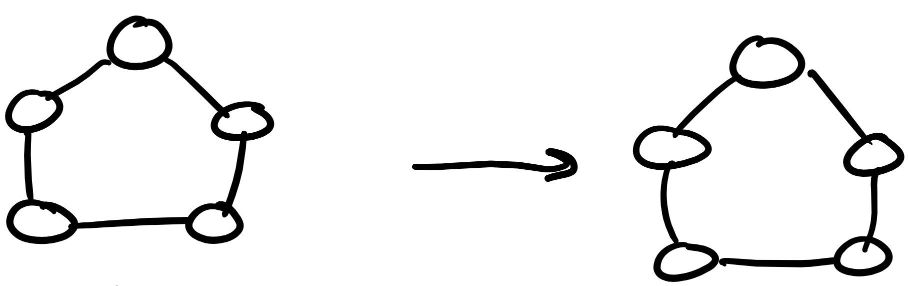
$\Rightarrow$ the closure of C_4 is K_4



K_6



closure is <u>NOT</u> Hamiltonian



C_5

closure of C_5 is not a clique but it is still Hamiltonian

sufficient condition:

sufficient condition:

If G's closure is a clique, then G is Hamiltonian

Q: Is the closure of G well-defined?

(we will always end up with the same closure result regardless of the order that we add edges)

Consider:

$e_1 e_2 \ldots e_i$ and $f_1 f_2 \ldots f_j$ are edges added to create closures of same $G \to G_e\ G_f$ (is $G_e = G_f$?)

→ since e_1 can be added for G_e, it can also be added as some f_k for G_f

→ if any e_ℓ depends on e_1, there is equivalently some f_m that is equivalently some f_m that depends on f_k, so f_m will also alway be added to G_s

=> apply this logic to all $e_1 \dots e_i$, we have an equivalent set of edges in $f_1 \dots f_j$

=> the closures are identical $\square$

Instead of explicitly constructing a closure of G

→ Look at the degree sequence of G to determine if the closure is a clique

Chvātal's condition

Consider G with degrees

Consider G with degrees

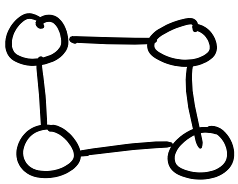
$d_1 \le d_2 \le \dots \le d_n$

if $i < \frac{n}{2}$ implies $d_i > i$

or $d_{n-i} \ge n-i$

$\Rightarrow$ the closure of G is a clique

(and G is Hamiltonian)



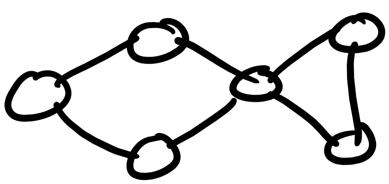
$S = 2\,2\,2\,2$

$i = 1\,2\,3\,4$

$n = 4$

$i = 1$

$d_1 = 2 > i = 1$ ✓



$S = 222224$

$i = 123456$

$n = 6$

$i = 1$

$d_1 = 2 > i = 1$ ✓

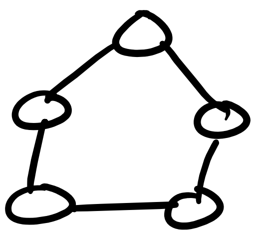
$i = 2$

$d_2 = 2 \not> i = 2$ ×

$d_{n-2} = d_4 = 2 \not\ge 4$ ×

condition doesn't hold for fish graph

Note: just because the condition doesn't hold, doesn't mean doesn't hold, doesn't mean the graph isn't Hamiltonian



$s = 22222$

$i = 12345$

$n = 5$

$i = 1$

$d_1 = 2 > 1$ ✓

$i = 2$

$d_2 = 2 \not> 2$ ×

$d_{n-2} = d_3 = 2 \not\geq n-2 = 3$ ×

condition doesn't hold

sufficient but not necessary