Maintaining Connectivity in Parallel Graph Partitioning

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Motivation

Partitioning graphs is an important preprocessing task in parallel scientific computing and graph analysis applications to balance work per-rank and minimize global communication. In many application, it is desirable for convergence and solution stability that each part within a partition remain fully **connected**. This work focuses on developing partitioning methods that ensure such connectivity.

Applications

General applications we consider:

- **Scientific computing** on fine meshes, which desires connectivity and multi-weight vertices
- Problem solving in biology, power grids, image processing, and many more [3]
- **Redistricting**, which requires connectivity, and can reduce *gerrymandering* issues in U.S. politics [2]

Background

Graph partitioning typically is the process of creating vertex-disjoint sets on a graph that minimize edge cut and constrain weight per-part. This project focuses on adding an additional constraint: **connectivity**. It is common for partitioning algorithms to not guarantee connected partitions, but it can be a highly desired feature. The work done in this project is built on an existing algorithm: **Partitioning using Label Propagation** (**PuLP**) [1].

Methods Explored for Maintaining Connectivity

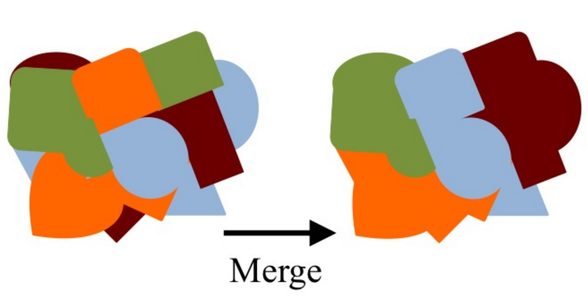
We are building upon an existing PuLP [1] algorithm, adding features and functionality to aid connectivity. The original algorithm has 2 inner iterations:

- **Balance** focuses on balancing the vertices between parts. Each vertex considers the *gain* to being in part P:

  $$gain(P) = n_P \cdot w_P$$

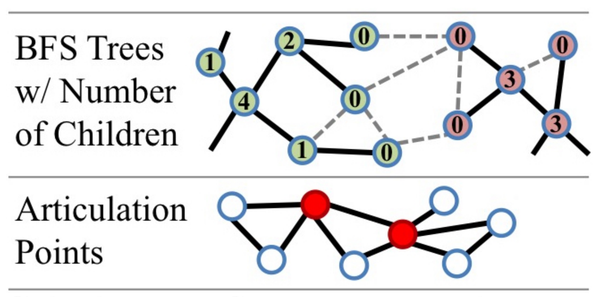
  n_P is the number of neighbors in P. w_P is how underweight P is.
- **Refine** works on minimizing the edge cut relative to each vertex.

Merge Small Components



Ideally, each part has one component. Every some number of iterations, merge all components that are not the largest in their part into a neighboring component.

Restrict Vertex Movement



Calculate BFS trees on each component, and only allow the **leaves** to move. Or, every so many vertex movements, calculate bi-connectivity, and prevent **articulation points** from moving.

Additional Metrics

Add a new metric to the *gain* equation by 2 methods:

(1) $gain(P) = (n_P + \boldsymbol{a_P}) \cdot w_P$

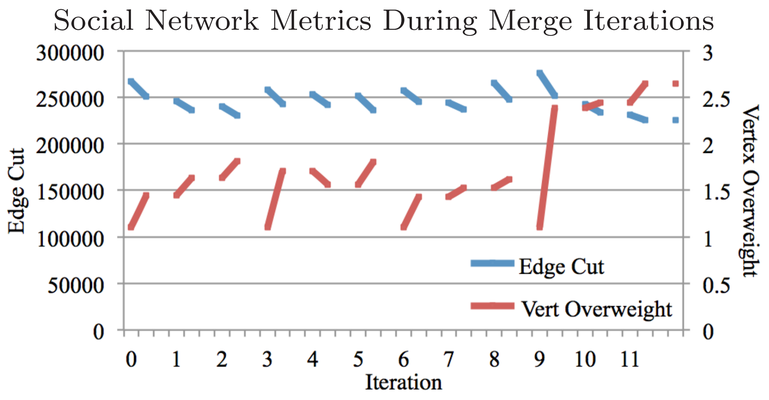
(2) $gain(P) = n_P \cdot w_P / \boldsymbol{a_P}$

$$a_P = \sum_{\substack{u \in N(v) \\ u \in P}} \frac{children(u)}{d(u)}$$

Where $N(v)$ is the neighborhood of v, $children(u)$ is the number of children u has in the component's BFS tree, and $d(u)$ is the degree of u.

Results

Merge Small Components



Merging Method: Here, we iteratively partition → merge disconnected components → partition. This merging procedure works well for regular graphs and meshes, while sacrificing some part balance and edge cut. But for more complex graphs, like a social network, part imbalance can be considerable.

Restrict Vertex Movement

C	Vertex Imbalance	Edge Cut	Small Comp. Count
0	1.5-2.5	>1,000K	0
1	1.1-1.2	800K	4,000
2	≤ 1.1	650K	5,000
∞	<1.1	150K	5,500

Leaves Only Method: We consider a BFS on each part from a central vertex. We then only allow vertices with C or less children to change parts.

R	10K	2.5K	1,250	320	80
Comp. Count	2,000	1,500	1,400	1,275	1,200

Articulation Point Method: Running on a road network, part biconnectivity is recalculated every R vertex updates. We restrict the movement of articulation/cut vertices for each part.

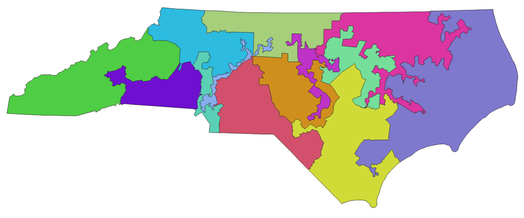
Additional Metrics

	Vertex Imb.	Edge Cut	Small Comps Count	Small Comps Size
Normal alg.	1.039	125.4K	6,100	22.17
Method (1)	1.040	125.5K	5,800	22.53
Method (2)	1.057	434.4K	14,400	13.94
Add Merging as a Final Step:				
Normal alg.	1.182	94.5K	0	N/A
Method (1)	1.174	95.3K	0	N/A
Method (2)	1.081	356K	0	N/A

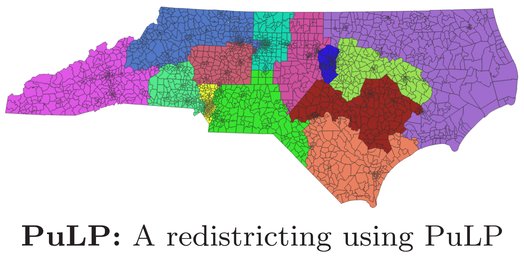
Using an Additional Metric: These numbers are from a web-crawl graph when adding the Average Number of Children metric, a_P when using the above gain equations of (1) and (2). Method (2) produces many components that are relatively small, allowing those components to be easily merged.

Application: Political Redistricting

Gerrymandering is the issue of politicians manipulating districts based on where voters live. Translating the redistricting problem to graph partitioning, **census blocks** are vertices, **districts** are parts, and edges represent shared borders. Previous work includes PEAR [2], which is an evolutionary algorithm. We've applied PuLP to counties in North Carolina, using population demographics as vertex weights and border lengths as edge weights.



Gerrymandering: North Carolina districts from 2013-2016.



PuLP: A redistricting using PuLP with small component merging.

Redistricting introduces new metrics that can be taken into consideration, such as compactness and competitiveness, as defined in [2]. **Compactness** is related to how well connected a district is and is defined as the area divided by perimeter squared. **Competitiveness** takes the numbers of voters in opposing parties to measure how evenly they're distributed in the districts. Other demographic data can also be considered.

References

[1] G. Slota, K. Madduri, S. Rajamanickam PULP: Scalable Multi-Objective Multi-Constraint Partitioning for Small-World Networks In *Proc. IEEE BigData Conf.*, 2014.

[2] Y. Y. Liu, W. K. T. Cho, S. Wang PEAR: a massively parallel evolutionary computation approach for political redistricting optimization and analysis. In *Swarm and Evolutionary Computation, 30, 78-92*, 2016.

[3] A. Buluc, H. Meyerhenke, I. Safro, P. Sanders, C. Schulz Recent Advances in Graph Partitioning In *Algorithm Engineering*, 2016.

Acknowledgements

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