Leximax and leximin extension rules for ranking sets as final outcomes with null alternatives

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Abstract
The objective of this study is to characterize the leximax and leximin extension rules for ranking sets as final outcomes. To rank any two subsets, we introduce null alternatives. We assume that each null alternative indicates ‘choosing not to choose each alternative.’ By adding null alternatives into each subset in which existing alternatives are not included, the cardinality of each transformed subset becomes equal to one of the set of all alternatives. Furthermore, all (null) alternatives in every transformed subset are rearranged in descending order. From these operations, we can rank all subsets lexicographically. The major result is axiomatization of the leximax and leximin extension rules by dominance axioms. Additionally, we clarify that the leximax and leximin extension rules satisfy monotonicity and extended independence.

1 Introduction
Three classes have been proposed in theories of ranking sets of alternatives (or objects) (see [2]). The first class is complete uncertainty (see [1, 3, 7, 10, 15, 17, 22]). In this class, probabilities are given for all alternatives in each subset, and each agent receives only one alternative chosen at random from a subset. The second class is opportunity sets (see [5, 6, 9, 11, 13, 14, 18, 19, 23]). In this class, each agent can choose only one alternative from a subset. The third class is sets as final outcomes (see [4, 12, 16, 20]). In this study, we analyse individual preference extension rules, not collective ones, for ranking sets as final outcomes.

From decision making in daily lives to policy making, we often choose a set of several alternatives based on a preference order of alternatives. Furthermore, we generally consider ‘compatibility’ of alternatives. For example, suppose that a soccer coach wants to hire two players, and there are three candidates: two forwards ($FW_A$ and $FW_B$), and one defender ($DF$). Additionally, for the coach, assume that a preference order is $FW_A$, $FW_B$, and $DF$, and $FW_A$ and $FW_B$ are incompatible. In this case, the coach will hire $FW_A$ and $DF$. However, in order to discuss how any two alternatives are (in)compatible, we need a reference point. We thus argue the simplest case with no compatibility of alternatives as the reference point. We then assume the following situation: an agent tries to plate some fruits at a reception. Note that the agent is assumed not to use a blender for consuming the fruits. Now, the goal is to lexicographically rank all subsets of the set of fruits. Particularly, we characterize the leximax and leximin extension rules.\footnote{We do not characterize a median-based rule, such as the Nitzan and Pattanaik [16] rule, because it belongs to a subgroup, including rules such as the maximax and maximin ones. Even if a leximedian rule can be defined, it would differ from the leximax and leximin rules, because we need one additional rule to rank pairs of both outsides of medians. Thus, it is excluded from this study.}

The main reference for this study is Bossert [4]. He characterized a group of lexicographic extension rules, including the leximax, leximin, median-based, and other lexicographical rules. However, subsets can be ranked if and only if they have the same cardinality, because...
Bossert [4] assumed a specific situation such as many-to-one matching with a given quota. To remove the restriction, we can use empty slots, which were introduced in Roth and Sotomayor [21]. Each empty slot is similar to an empty set or outside option, and assumed to be added to each subset whose cardinality is smaller than a given quota. However, we consider situations in which there is no fixed quota to apply extension rules to more general choice theories.

We then introduce null alternatives, which are assumed to indicate ‘choosing not to choose alternatives.’ If each alternative is not included in each subset, we add its null alternative to the subset, and rearrange all (null) alternatives in descending order. Following these operations, all subsets can be ranked lexicographically. Additionally, each alternative is defined as (un)desirable if and only if it is strictly better (worse) than its null alternative, and neutral if and only if it and its null alternative are indifferent. For example, suppose that \{a, b, c, d\} is the set of all alternatives, and for an agent, \(a\) and \(b\) are indifferent, \(b\) is desirable, all null alternatives are indifferent, \(c\) is undesirable, and \(c\) is strictly better than \(d\). Additionally, assume that \(n_a\) denotes a null alternative of \(a\). If the agent forcibly ranks \{a, c, d\} and \{b, d\} based on the leximax criteria, \{a, c, d\} is strictly better than \{b, d\}. However, this is non-intuitive because the third and second alternatives of \{a, c, d\} and \{b, d\} are the same, and \(c\) is undesirable. By adding null alternatives and rearranging them, \{a, c, d\} and \{b, d\} are transformed into \{a, n_a, c, d\} and \{b, n_a, n_c, d\}, respectively. We thus obtain that \{b, d\} is strictly better than \{a, c, d\}.

In this study, we do not assume a property called null-indifference. This requires that all null alternatives are indifferent, such as empty slots. The following example illustrates an advantage in relaxing null-indifference. Suppose that \{a, b, c\} is the set of all alternatives, and an agent strictly prefers \(a\) to \(b\), likes \(a\) and \(b\) intermediately, and hates \(c\) enormously. If we assume null-indifference, \{a, c\} and \{b\} will be transformed into \{a, n_b, c\} and \{b, n_a, n_c\}, respectively, and the agent will strictly prefer \{a, c\} to \{b\} based on the leximax criteria. However, this is non-intuitive because the agent hates \(c\) enormously. Now, if the agent is allowed to strictly prefer \(n_c\) to \(a\), the two subsets will be transformed into \{a, n_b, c\} and \{n_c, b, n_a\}, and the agent will strictly prefer \{b\} to \{a, c\}. Note that we need a weaker property than null-indifference to make a preference order of alternatives equivalent to one of their singleton sets. We thus assume asymmetry of desirability. This requires that a preference order of any two alternatives and one of their null alternatives are opposite.

The major result is characterization of the leximax and leximin extension rules by dominance axioms. Furthermore, we obtain that the leximax and leximin extension rules satisfy monotonicity and extended independence.

The remainder of this paper is structured as follows. Section 2 reports our notations and definitions. Section 3 discusses the necessity of asymmetry of desirability. Section 4 introduces dominance axioms for preference relations on the power set. Section 5 axiomatizes the leximax and leximin extension rules. Additionally, the section introduces monotonicity and independence axioms to clarify more necessary conditions for deriving the extension rules. Finally, our conclusions are provided in Section 6.

2 Preliminary

Let \(X\) be the finite set of all alternatives with cardinality \(|X| \geq 2\). The power set of \(X\) is denoted by \(\mathcal{P}\). Let \(N = \cup_{a \in X} \{n_a\}\) be the finite set of null alternatives such that ‘choosing not to choose \(a\)’ is equivalent to ‘choosing \(n_a\’\) for all \(a \in X\). Furthermore, each subset of \(N\) is denoted by \(N_A = \cup_{a \in A} \{n_a\}\), corresponding to each subset \(A \in \mathcal{P}\). A preference relation on \(X \cup N\) is assumed to be a complete preordering denoted by \(R \in \mathcal{R}\), where \(\mathcal{R}\) is the set of all preference relations on \(X \cup N\). The asymmetric and symmetric components
are denoted by \( P \) and \( I \), respectively. We define (un)desirability of alternatives as follows: \( a \) is (un)desirable if and only if \( a \in P \) (\( n \in P \)), and neutral if and only if \( a \in I \) for each \( a \in X \). Additionally, let \( \mathcal{R} \) be a preference relation on \( X \), where \( \mathcal{R} \) is the set of all preference relations on \( X \). The asymmetric and symmetric components are denoted by \( \mathcal{P} \) and \( \mathcal{I} \), respectively.

Next, we discuss a method for ranking all subsets lexicographically. In Bossert [4], subsets might not be ranked if they have different cardinalities. For example, when \( X = \{a, b, c\} \), \( \{a, b\} \) and \( \{c\} \) cannot be ranked if \( a \in \mathcal{I} \). Several methods are used to solve this problem. Roth and Sotomayor [21] assumed a situation similar to a college admissions problem, and introduced the concept of empty slots. Empty slots are added to each subset whose cardinality is smaller than a given quota. However, we also consider problems in more general choice theories. Furthermore, empty slots are the same with null alternatives assumed to satisfy null-indifference, that is, \( n_a I n_b \) for all \( a, b \in X \). This restricts the scope of considerable situations, because \( n_a I n_b \) might be non-intuitive if the agent hates \( a \) and likes \( b \). Thus, we do not assume null-indifference in this study.

We then introduce transformed subsets by adding null alternatives. For each \( A \in \mathcal{X}^* \), let \( f_A : X \to A \cup (X \setminus N) \) be a bijection such that \( f_A(a) = a \) when \( a \in A \), and \( f_A(a) = n_a \) when \( a \not\in A \) for all \( a \in X \). Let \( A^* = \cup_{A \in \mathcal{X}} f_A(A) \) be the transformed subset of \( A \), and \( \mathcal{X}^* = \cup_{A \in \mathcal{X}} \{A^*\} \) be the transformed power set of \( X \). Furthermore, all (null) alternatives are assumed to be rearranged in descending order: for all \( A^* = \{a_1^*, a_2^*, \ldots, a_{|X|}^*\} \in \mathcal{X}^* \), \( a_i^* Ra_{i+1}^* \) for all \( i \in \{1, 2, \ldots, |X| - 1\} \). Thus, all subsets can be ranked even if they have different cardinalities by using the transformed subsets.

Finally, the lexicmax and lexicmin extension rules are defined in the following manner:

**Definition 1.** Leximax extension rule \( \hat{R}_{\text{leximax}}: \forall A, B \in \mathcal{X} \),

\[
\hat{R}_{\text{leximax}} A \iff \exists i \in \{1, 2, \ldots, |X|\} \text{ s.t. } a_i^* Pb_i^* \land a_i^* P b_j^* \forall j < i;
\]

\[
\hat{R}_{\text{leximax}} A \iff a_i^* Pb_i^* \forall i \in \{1, 2, \ldots, |X|\}.
\]

**Definition 2.** Leximin extension rule \( \hat{R}_{\text{leximin}}: \forall A, B \in \mathcal{X} \),

\[
\hat{R}_{\text{leximin}} A \iff \exists i \in \{1, 2, \ldots, |X|\} \text{ s.t. } a_i^* Pb_i^* \land a_i^* P b_j^* \forall j > i;
\]

\[
\hat{R}_{\text{leximin}} A \iff a_i^* Pb_i^* \forall i \in \{1, 2, \ldots, |X|\}.
\]

By using \( \hat{R}_{\text{leximax}} \) and \( \hat{R}_{\text{leximin}} \), we can avoid some non-intuitive preference orders, as stated earlier in Section 1. Suppose that \( X = \{a, b, c, d\} \), \( a \in P \), \( n \in P \), \( c \in P \), and \( b \in \mathcal{I} \). Without the transformed subsets, they are forcibly ranked according to the lexicimax criteria as follows: \( AB \). However, the third and second alternatives of \( A \) and \( B \) are \( d \) and \( n \), respectively, so we can obtain that \( B \hat{R}_{\text{leximax}} A \).

However, \( \hat{R}_{\text{leximax}} \) and \( \hat{R}_{\text{leximin}} \) have a serious problem. \( \hat{R}_{\text{leximax}} \) and \( \hat{R}_{\text{leximin}} \) should satisfy a condition in order to provide consistent preference orders of singleton sets based on \( R \), namely, extensibility. This requires that a preference order of any two alternatives and one of their singleton sets are the same.

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2There are two similar concepts: an outside option and a threshold in the preference approval voting rule (see [8]). Suppose that \( X = \{a, b, c\} \). The outside option is often denoted by \( \emptyset \), and \( a \in P \) indicates that \( a \) and \( b \) are desirable and \( c \) is undesirable, where \( P' \) is the asymmetric component of \( R \). Similarly, the threshold is denoted by \( \cdot \), and \( ab \) is equivalent to \( ab \in P \). However, both of them are not appropriate to rank all subsets because we cannot frame the cardinalities of any two subsets using them.

3Extensibility was called an extension rule in related fields, such as complete uncertainty and opportunity sets (see [2]).
In cases (i) and (ii), aRb 
P_n bPa. In the following example, \( \hat{R}_{\text{max}} \) and \( \hat{R}_{\text{min}} \) violate extensibility: \( X = \{a, b\} \) and \( n_a PaPbPn_b \). Even if \( aPb \), \( \{b\} \hat{R}_{\text{max}} \{a\} \) and \( \{a\} \hat{R}_{\text{min}} \{b\} \) because \( \{a\}^* = \{a, n_b\} \) and \( \{b\}^* = \{n_a, b\} \). However, \( n_a PaPbPn_b \) is non-intuitive because \( b \) is desirable and \( a \) is undesirable.

### 3 Requirements for null alternatives

To solve the above problem, we introduce properties for \( R \) and check whether they make \( \hat{R}_{\text{max}} \) and \( \hat{R}_{\text{min}} \) satisfy extensibility.

First, consistency of desirability requires that every desirable alternative is strictly better than every neutral or undesirable alternative, every neutral alternative is strictly better than every undesirable alternative, and any two neutral alternatives are indifferent.

**Consistency of desirability:** \( \forall a, b \in X, [aPa_b \land n_b Rb (a) \lor [aIn_a \land n_b Pb]] \Rightarrow aPb; [aIn_a \land bIn_b] \Rightarrow aIb. \)

Consistency of desirability is suitable to the meaning of null alternatives, but not enough to imply extensibility. Suppose that \( X = \{a, b\} \) and \( n_a Pn_bPaIb \). This preference order does not violate consistency of desirability. However, \( \{b\} \hat{R}_{\text{max}} \{a\} \) and \( \{a\} \hat{R}_{\text{min}} \{b\} \) even if \( aIb \). We thus need to introduce a stronger property than consistency of desirability.

Now, we introduce asymmetry of desirability. This requires that a preference order of any two alternatives and one of their null alternatives are opposite.

**Asymmetry of desirability:** \( \forall a, b \in X, aRb \Rightarrow n_bR_a. \)

From Proposition 1, asymmetry of desirability implies consistency of desirability.

**Proposition 1.** \( R \) satisfies consistency of desirability if \( R \) satisfies asymmetry of desirability.

**Proof.** Let \( R \) satisfy asymmetry of desirability. By way of contradiction, take any two alternatives \( a, b \in X \) and assume that \( bRa \) when (i) \( aPa_n a \) and \( n_b Rb \), or (ii) \( aIn_a \) and \( n_b Pb \). By asymmetry of desirability, \( bRa \) implies that \( n_a Rn_b \). We thus obtain \( aPb \) by transitivity in both cases (i) and (ii), but that is a contradiction.

Next, assume that (iii) \( aPb \) or (iv) \( bPa \) when \( aIn_a \) and \( bIn_b \). By asymmetry of desirability, \( aPb \) and \( bPa \) implies \( n_b Rn_a \) and \( n_a Rn_b \), respectively. Thus, we obtain \( bRa \) in case (iii) and \( aRb \) in case (iv) by transitivity. These results are contradictions.

Thus, \( R \) satisfies consistency of desirability if \( R \) satisfies asymmetry of desirability. \( \square \)

From Lemma 1, asymmetry of desirability is one of the sufficient conditions for \( \hat{R}_{\text{max}} \) and \( \hat{R}_{\text{min}} \) to satisfy extensibility.

**Lemma 1.** \( \hat{R}_{\text{max}} \) and \( \hat{R}_{\text{min}} \) satisfy extensibility if \( R \) satisfies asymmetry of desirability.

**Proof.** Assume that \( R \) satisfies asymmetry of desirability. First, we prove that \( \{a\} \hat{R}_{\text{max}} \{b\} \) implies \( aRb \) for all \( a, b \in X \). Take any two alternatives \( a, b \in X \) such that \( \{a\} \hat{R}_{\text{max}} \{b\} \).

The difference between \( \{a\}^* \) and \( \{b\}^* \) is that \( a, n_b \not\in \{b\}^* \) and \( n_a, b \not\in \{a\}^* \). By Definition 1 and asymmetry of desirability, \( \{a\} \hat{R}_{\text{max}} \{b\} \) if and only if

(i) \( [aRa_n \land n_a Rb] \Rightarrow [aPa_n \lor [aIn_a \land n_b Pb]]; \)

(ii) \( [aRa_n \land n_b Rb] \Rightarrow aRb; \)

(iii) \( [n_b Ra \land n_a Rb] \Rightarrow [n_b Pa \lor [n_b In_a \land aRb]]; \) and

(iv) \( [n_a Ra \land n_b Rb] \Rightarrow [n_a Pb \lor [n_a In_b \land aRn_a]]. \)

In cases (i) and (ii), \( aRb \) holds true. In cases (iii) and (iv), by way of contradiction, suppose that \( bPa \), implying \( n_a Rn_b \) by asymmetry of desirability. However, the assumption
contradicts \( n_b \) \( P_n a \) in both cases, and \( aRb \) in Case (iii). Thus, \( aRb \) holds true in all the four cases.

Next, we prove that \( aRb \) implies \( \{a\} R_{lmax} \{b\} \) for all \( a, b \in X \). Take any two alternatives \( a, b \in X \) such that \( aRb \). By asymmetry of desirability, \( aRb \) implies \( n_b R_{m_a} \). We then obtain all the four results (i)-(iv), in other words, \( \{a\} R_{lmax} \{b\} \).

Thus, \( R_{lmax} \) satisfies extensibility if \( R \) satisfies asymmetry of desirability. Similarly, \( R_{lmin} \) satisfies extensibility if \( R \) satisfies asymmetry of desirability. \( \square \)

However, extensibility does not imply asymmetry of desirability. For instance, suppose that \( X = \{a, b, c\} \) and \( aPn_a, Pn_a PbPc \) for an agent. In this case, \( aPb, \{a\} R_{lmax} \{b\} \), and \( \{a\} R_{lmin} \{b\} \), but \( aPb \) does not imply \( n_b R_{m_a} \). Thus, we should discuss the strength of asymmetry of desirability. First, take any two alternatives \( a, b \in X \). In total, there are seventy-five preference orders of \( a, b, n_a, n_b \in X \cup N \) since \( R \) is a complete preordering on \( X \cup N \). In forty-five of the seventy-five orders, both \( R_{lmax} \) and \( R_{lmin} \) satisfy extensibility. Furthermore, in thirty-nine of the forty-five orders, \( R \) satisfies asymmetry of desirability.

If the agent has one of the following six of the forty-five orders, both \( R_{lmax} \) and \( R_{lmin} \) satisfy extensibility, but violate asymmetry of desirability: \( aPn_a Pb, aPn_a Pb, bPn_b Pn_b, aPn_a Pb, bPn_b Pn_b \), \( bPn_b \), \( aPn_a \), \( bPn_b \), \( aPn_a Pb, bPn_b Pn_b \). Thus, asymmetry of desirability is not the necessary condition to make only \( R_{lmax} \) and \( R_{lmin} \) satisfy extensibility. However, there are more lexicographic extension rules, such as the median-based and leximedian ones. From the discussion, asymmetry of desirability might not be strong enough to consider extensibility. Additionally, only in thirteen of the forty-five orders, \( R \) satisfies null-indifference. Thus, we can relax the strong restriction of null-indifference and obtain consistent preference orders of singleton sets based on \( R \) by using asymmetry of desirability.

We then suppose that \( R^t \) denotes \( R \) satisfying asymmetry of desirability. Thus, \( R_{lmax} \) and \( R_{lmin} \) are redefined using \( R^t \) instead of \( R \) as follows:

**Definition 3.** Leximax extension rule \( R_{lmax}^t \): \( \forall A, B \in \mathcal{X} \),

\[
A R_{lmax}^t B \Leftrightarrow \exists i \in \{1, 2, ..., |X|\} \text{ s.t. } a_i^* P^1 b_i^* \ \& \ \forall j < i;
\]

\[
A R_{lmin}^t B \Leftrightarrow \exists i \in \{1, 2, ..., |X|\} \text{ s.t. } a_i^* P^1 b_i^* \ \& \ \forall j > i;
\]

**Definition 4.** Leximin extension rule \( R_{lmin}^t \): \( \forall A, B \in \mathcal{X} \),

\[
A R_{lmin}^t B \Leftrightarrow \exists i \in \{1, 2, ..., |X|\} \text{ s.t. } a_i^* P^1 b_i^* \ \& \ \forall j > i;
\]

Finally, we discuss an advantage in employing \( R^t \). Suppose that \( \{a, b, c\} \) and \( aPbPc \) for an agent. Furthermore, assume that the agent likes \( a \) and \( b \) intermediately, but hates \( c \) enormously. We then consider \( \{a, c\} \) and \( \{b\} \). If \( R \) satisfies null-indifference, \( \{a, c\}^* = \{a, n_b, c\} \), \( \{b\}^* = \{b, n_a, n_c\} \), and \( \{a, c\} R_{lmax} \{b\} \). However, from the setting, \( aPn_a \) is non-intuitive. This is a serious disadvantage in using the leximax criteria. If we allow that \( n_c P^1 a \), then \( \{a, c\}^* = \{a, n_b, c\} \), \( \{b\}^* = \{n_c, b, n_a\} \), and we will obtain that \( \{b\} R_{lmax}^t \{a, c\} \). Thus, we can express a certain degree of desirability, and adapt a part of the leximin (leximax) criteria in the leximax (leximin) extension rule by using \( R^t \). We thus discuss characterizations of \( R_{lmax}^t \) and \( R_{lmin}^t \) hereafter.
4 Axioms

Bossert [4] introduced two axioms with a fixed cardinality of subsets to characterize the group of lexicographic extension rules: *responsiveness*\(^4\) and *neutrality*.\(^5\) However, we characterize only the *leximax* and *leximin* extension rules separately. Thus, we employ another approach to characterize \(R_{\text{leximax}}^l\) and \(R_{\text{leximin}}^l\) by the following three axioms.

First, *indifference dominance* requires that \(AIB\) for all \(A, B \in \mathcal{X}\) if all elements in \(A^*\) and \(B^*\) are indifferent for every rank.

**Indifference dominance:** \(\forall A, B \in \mathcal{X}, \{a_j^*b_j^* \mid i \in \{1, 2, \ldots, |X|\}\} \Rightarrow AIB.\)

The second (third) axioms is *prior (posterior) strict dominance*. This requires that \(APB\) for all \(A, B \in \mathcal{X}\) if each element of \(A^*\) dominates one of \(B^*\) for every rank, and there is at least one strict preference order in certain prior (posterior) parts of them. Thus, it might be said that they are partial conditions of strict dominance.\(^6\)

**Prior strict dominance:** \(\forall A, B \in \mathcal{X}, \{3k \in \{1, 2, \ldots, |X|\} \text{ s.t. } a_k^*Rb_k^* \forall i \in \{1, 2, \ldots, k\}\} \land \exists j \in \{1, 2, \ldots, k\} \text{ s.t. } a_j^*Pb_j^* \Rightarrow APB.\)

**Posterior strict dominance:** \(\forall A, B \in \mathcal{X}, \{3l \in \{1, 2, \ldots, |X|\} \text{ s.t. } a_l^*Rb_l^* \forall i \in \{l, l + 1, \ldots, |X|\}\} \land \exists j \in \{l, l + 1, \ldots, |X|\} \text{ s.t. } a_l^*Pb_l^* \Rightarrow APB.\)

5 Characterizations

First, from Lemma 2, \(\tilde{R}_{\text{leximax}}^l\) and \(\tilde{R}_{\text{leximin}}^l\) are complete preorderings on \(\mathcal{X}\).

**Lemma 2.** \(\tilde{R}_{\text{leximax}}^l\) and \(\tilde{R}_{\text{leximin}}^l\) satisfy reflexivity, completeness, and transitivity.

**Proof.** From Definitions 3 and 4, \(\tilde{R}_{\text{leximax}}^l\) satisfies reflexivity and completeness. Thus, we prove that \(R_{\text{leximax}}^l\) satisfies transitivity.

For all \(A, B, C \in \mathcal{X}\), \(AP_{\text{leximax}}^l\) and \(BP_{\text{leximax}}^l\) if and only if there exist \(i, k \in \{1, 2, \ldots, |X|\}\) such that \(a_j^*Pb_j^*\) and \(a_j^*Pb_j^*\) for all \(j < i, k \leq j \leq k < i\). Then, \(k < i\) implies that \(a_k^*Pb_k^*\) \(a_k^*Pb_k^*\) for all \(j < k, k > i\) implies that \(a_k^*Pb_k^*\) \(a_k^*Pb_k^*\) for all \(j > k\). Then, \(a_k^*Pb_k^*\) \(a_k^*Pb_k^*\) for all \(j < k\). Then, \(a_k^*Pb_k^*\) \(a_k^*Pb_k^*\) for all \(j > k\). Then, \(a_k^*Pb_k^*\) \(a_k^*Pb_k^*\) for all \(j > k\). Then, \(a_k^*Pb_k^*\) \(a_k^*Pb_k^*\) for all \(j > k\).

Next, \(AP_{\text{leximax}}^l\) and \(BP_{\text{leximax}}^l\) if and only if there exists \(i \in \{1, 2, \ldots, |X|\}\) such that \(a_j^*Pb_j^*\) \(a_j^*Pb_j^*\) for all \(j < i\) and \(a_j^*Pb_j^*\) \(a_j^*Pb_j^*\) for all \(k \leq j \leq k < i\). Then, \(a_j^*Pb_j^*\) \(a_j^*Pb_j^*\) for all \(j < i\) from transitivity of \(R_{\text{leximax}}^l\). Thus, if \(AP_{\text{leximax}}^l\) and \(BP_{\text{leximax}}^l\), then \(AP_{\text{leximax}}^l\) for all \(A, B, C \in \mathcal{X}\). Similarly, if \(AP_{\text{leximax}}^l\) and \(BP_{\text{leximax}}^l\), then \(AP_{\text{leximax}}^l\) for all \(A, B, C \in \mathcal{X}\).

Finally, \(AP_{\text{leximax}}^l\) and \(BP_{\text{leximax}}^l\) if and only if \(a_j^*Pb_j^*\) \(a_j^*Pb_j^*\) for all \(i \in \{1, 2, \ldots, |X|\}\). Thus, if \(AP_{\text{leximax}}^l\) and \(BP_{\text{leximax}}^l\), then \(AP_{\text{leximax}}^l\) for all \(A, B, C \in \mathcal{X}\).

From these results, \(R_{\text{leximax}}^l\) satisfies transitivity. Similarly, \(R_{\text{leximin}}^l\) satisfies reflexivity, completeness, and transitivity.

The major result is Theorem 1 that describes the necessary and sufficient conditions to derive \(R_{\text{leximax}}^l\) and \(R_{\text{leximin}}^l\).

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\(^4\)\(\forall A \in \mathcal{X}, \{A \subseteq X \mid |A| = k\}, \forall b \in X, y \in X \setminus A, b \notin c \Rightarrow A \setminus b \subset c,\)

\(^5\)\(\forall A, B \in \mathcal{X}, \forall \sigma : X \to X, \sigma(R) \subset \sigma(a) \subset \sigma(R(b)) \forall A \subset B \Rightarrow |ARB| \subset \sigma(a) \subset A \setminus \sigma(b),\)

\(^6\)\(\forall A, B \in \mathcal{X}, \{[a_j^*Pb_j^*] \forall i \in \{1, 2, \ldots, |X|\}\} \land \exists j \in \{1, 2, \ldots, |X|\} \text{ s.t. } a_j^*Pb_j^* \Rightarrow APB.\)
Theorem 1. \( \hat{R} = \hat{R}_{\text{max}}^i \) (\( \hat{R} = \hat{R}_{\text{min}}^i \)) if and only if \( R = R^i \) and \( \hat{R} \) satisfies reflexivity, completeness, transitivity, indifference dominance, and prior (posterior) strict dominance.

Proof. From Lemma 2 and Definition 3, \( R = R^i \), \( \hat{R}_{\text{max}}^i \) is a complete preordering on \( \mathcal{X} \), and trivially satisfies indifference dominance and prior strict dominance.

We then prove that the following two propositions hold true if \( R = R^i \) and \( \hat{R} \) satisfies reflexivity, completeness, transitivity, indifference dominance, and prior strict dominance:

(i) \( AIB \iff a^*_i P^i b^*_i \forall i \in \{1, 2, \ldots, |X|\} \);

(ii) \( APB \iff \exists i \in \{1, 2, \ldots, |X|\} \text{ s.t. } a^*_i P^i b^*_i \land a^*_j P^i b^*_j \forall j < i \).

The `if' parts of (i) and (ii): They are trivial from \( R = R^i \), indifference dominance, and prior strict dominance.

The `only if' part of (i): By way of contradiction, let \( AIB \) imply the existence of some \( i \in \{1, 2, \ldots, |X|\} \) such that \( a^*_i P^i b^*_i \) or \( b^*_i P^i a^*_i \). Suppose that \( i' \) is the argument of the minimum of \( i \) such that \( a^*_i P^i b^*_i \) or \( b^*_i P^i a^*_i \). By prior strict dominance, \( APB \) if \( a^*_i P^i b^*_i \), and \( BPA \) if \( b^*_i P^i a^*_i \). They contradict \( AIB \) because \( R \) is a complete preordering on \( \mathcal{X} \).

The `only if' part of (ii): By way of contradiction, let \( APB \) imply that \( a^*_i P^i b^*_i \) for all \( i \in \{1, 2, \ldots, |X|\} \) or there exists \( i \in \{1, 2, \ldots, |X|\} \) such that \( b^*_i P^i a^*_i \) and \( a^*_j P^i b^*_j \) for all \( j < i \).

Each case respectively implies that \( AIB \) or \( BPA \) by the `if' parts of (i) and (ii). They contradict \( APB \) since \( R \) is a complete preordering on \( \mathcal{X} \).

Thus, \( \hat{R} = \hat{R}_{\text{max}}^i \) if and only if \( R = R^i \) and \( \hat{R} \) is a complete preordering satisfying indifference dominance and prior strict dominance. Similarly, \( \hat{R} = \hat{R}_{\text{min}}^i \) if and only if \( R = R^i \) and \( \hat{R} \) is a complete preordering satisfying indifference dominance and posterior strict dominance.

Theorem 1 shows the axiomatization of \( \hat{R}_{\text{max}}^i \) and \( \hat{R}_{\text{min}}^i \). However, they seem predictable following indifference dominance, prior strict dominance, and posterior strict dominance. We need these critical axioms to characterize \( \hat{R}_{\text{max}}^i \) and \( \hat{R}_{\text{min}}^i \) because lexicographical comparison methods have restrictions such that we must begin to compare from the best or worst (null) alternatives and stop comparing alternatives if we find a strict preference order. Thus, it is crucial to discuss whether \( \hat{R}_{\text{max}}^i \) and \( \hat{R}_{\text{min}}^i \) satisfy other axioms. Indeed, some axioms should be satisfied by them, because we assume no compatibility of alternatives in this study. We then introduce additional axioms and show that \( \hat{R}_{\text{max}}^i \) and \( \hat{R}_{\text{min}}^i \) satisfy them.

First, monotonicity requires that a rank of each subset increases (decreases) by adding every (un)desirable alternative, but does not change by adding any neutral alternative. Note that its definition is based on the relationship between alternatives and their null alternatives.

Monotonicity: \( \forall A \in \mathcal{X}, \forall a \in X \setminus A, a R^i a \iff A \cup \{a\} R^i A \).

Next, extended independence requires that a preference order of any two subsets is not affected by adding a disjoint subset to both subsets.

Extended independence: \( \forall A, B \in \mathcal{X}, \forall C \subseteq X \setminus (A \cup B), A R^i B \iff A \cup C R^i B \cup C \).

Note that extended independence implies independence\(^7\), extended responsiveness\(^8\), and other weaker related axioms. Extended monotonicity\(^9\) is also defined as one of weaker

\(^7\)\( \forall a, b \in X, \forall C \subseteq X \setminus \{a, b\}, a R b \iff \{a\} \cup C R(b) \cup C \).

\(^8\)\( \forall A \in \mathcal{X}, B \subseteq A, \forall C \subseteq X \setminus A, B R^i C \iff A R^i (A \setminus B) \cup C \).

\(^9\)\( \forall A \in \mathcal{X}, \forall B \subseteq X \setminus A, B R^i A \iff \emptyset \cup B R^i A \).
axioms than extended independence. However, its definition is based on the relationship between subsets and an empty set, not their null subsets. Additionally, we cannot consider another monotonicity\(^{10}\) that is weaker than extended monotonicity because \(R\) is defined on \(X \cup N\), which does not include an empty set. Thus, we consider monotonicity and extended independence separately.

Finally, Theorem 2 shows that \(\hat{R}_{\text{imax}}^t\) and \(\hat{R}_{\text{imin}}^t\) satisfy the above axioms.

**Theorem 2.** \(\hat{R}_{\text{imax}}^t\) and \(\hat{R}_{\text{imin}}^t\) satisfy monotonicity and extended independence.

*Proof.* First, we prove that \(\hat{R}_{\text{imax}}^t\) satisfies monotonicity. Suppose that \(a_i^* = a_a \in A^*\) and \(B = A \cup \{a\}\). Then, \(a_iR^t a_a\) if and only if \(a = b_j^*\), where \(1 \leq j \leq i\). In this case, \(b_k^* = a_k^*\) for all \(k \in \{1, 2, ..., j - 1, i + 1, ..., |X|\}\) and \(b_{l+1}^* = a_i^*\) for all \(l \in \{j, 2, ..., i\}\). Thus, we obtain that \(a_iR^t a_a\) if and only if \(a_iR^t a_i^*\), which implies \(BR_{\text{imax}}^t A\).

Second, we prove that \(\hat{R}_{\text{imin}}^t\) satisfies extended independence. Take any three subsets: \(A, B \in \mathcal{X}^*, \text{ and } C \subseteq X \setminus (A \cup B)\). From Definition 3, \(A\hat{R}_{\text{imax}}^t B\) if and only if \(a_i^*P^ib_i^*\), \(i \in \{1, 2, ..., |X|\}\), and \(a_i^*P^ib_i^*\) for all \(j < i\). Suppose that there are \(k\) alternatives in \(\bar{C} = \{c_1, c_2, ..., c_k\}\) such that \(a_i^*P^ib_i^*_j\) for all \(j < i\). Then, \(k = 0\) implies that \(A \cup B = A\hat{C}B \cup C\) because \(a_i^*P^ib_i^*_j\) and transitivity of \(R^t\). Furthermore, \(k \geq 1\) also implies that \(A \cup C\hat{R}_{\text{imax}}^t B \cup C\). From these results, we obtain that \(A\hat{R}_{\text{imax}}^t B\) if and only if \(A \cup C\hat{R}_{\text{imax}}^t B \cup C\). Next, from Definition 3, \(A\hat{R}_{\text{imin}}^t B\) if and only if \(a_i^*P^ib_i^*_j\) for all \(j \in \{1, 2, ..., |X|\}\). In this case, the positions of \(c_k\) in \(C\) in \((A \cup C)^+\) and \((B \cup C)^+\) have to be the same for all \(k \in \{1, 2, ..., |C|\}\). Finally, we obtain that \(A\hat{R}_{\text{imin}}^t B\) if and only if \(A \cup C\hat{R}_{\text{imin}}^t B \cup C\).

Similarly, \(\hat{R}_{\text{imin}}^t\) satisfies monotonicity and extended independence. \(\Box\)

### 6 Conclusion

In this paper, we introduce null alternatives to frame the cardinalities of all subsets and rank them. Unlike empty slots, strict preference orders of null alternatives are allowed. According to the above framework, we define the leximax and leximin extension rules on \(\mathcal{X}^*\).

However, if there is no requirement for null alternatives, a complete preordering \(R\) and the extension rules do not satisfy extensibility. Thus, we employ \(R^t\) on \(X \cup N\), which is a complete preordering satisfying asymmetry of desirability, to obtain consistent preference orders of singleton sets based on ones of alternatives.

We then axiomatize the extension rules as follows: A preference relation \(R\) on \(\mathcal{X}\) is the leximax extension rule if and only if \(R^t\) is equal to \(R^t\) and \(\hat{R}\) is a complete preordering satisfying indifference dominance and prior strict dominance. Furthermore, a preference relation \(\hat{R}\) on \(\mathcal{X}\) is the leximin extension rule if and only if \(R\) is equal to \(R^t\) and \(\hat{R}\) is a complete preordering satisfying indifference dominance and posterior strict dominance.

Additionally, we find that the leximax and leximin extension rules satisfy monotonicity and extended independence, which should be satisfied when there is no compatibility of alternatives.

Lastly, by using null alternatives, we can more generally apply the leximax and leximin extension rules on the power set to various fields of choice theories when we must use ordinal extension rules on the power set, and consider a certain degree of desirability.

\(^{10}\forall a \in X, \forall B \subseteq X \setminus \{a\}, aR^tB \leftrightarrow \{a\} \cup B \hat{R} B\), where \(R^t\) is a preference relation on \(X \cup N \cup \{\emptyset\}\).
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