Assigning indivisible and categorized items

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In this paper, we study the problem of assigning indivisible items under the following constraints: 1) each item belongs to one of the p categories, 2) each agent is required to get exactly one item from each category, 3) no free disposal, and 4) no monetary transfer.

We characterize serial dictatorships by a minimal set of 3 properties: *strategy-proofness*, *non-bossiness*, and *category-wise neutrality*. Then, we analyze a natural extension of serial dictatorships called *sequential allocation mechanisms*, which allocate the items in multiple stages according to a given order over all (agent,category) pairs, so that in each stage, the designated agent chooses an item from the designated category. We focus on the cases where each agent is either *optimistic* or *pessimistic*, and for any sequential allocation mechanism, we characterize a tight lower bound on the rank of the allocated bundle for each agent. This characterization shows us to study the worst-case efficiency loss and thus help us choose an optimal sequential allocation in this regard.

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1. INTRODUCTION

Suppose you are organizing a conference and you want to allocate catered lunch to n agents (conference attendees). The lunch consists of three categories of foods: wraps, deserts, and bottles of drinks. Each category contains exactly n indivisible and heterogeneous items,[1] and each agent must get exactly one item in each category. Agents may have different preferences over the (wrap,desert,drink) combinations. How would you allocate these items?

Many other allocation problems share this categorized feature: allocating rooms, tables, chairs, and computers among a group of students; allocating tent, car, GPS among a group of families for camping out; allocating time slots and topics for seminar paper presentation.

These are a special case of a more general (and much harder) problem that has been extensively studied by researchers in economics and artificial intelligence. Suppose there are m indivisible items and n agents. The agents may have different private preferences over bundles of items, and we want to design a *mechanism* to allocate these items to agents without monetary transfer. This problem in its most general form is too hard due to high complexity of preference representation and communication. Therefore, most previous work focused on designing sensible mechanisms with various constraints on agents' preference structures.

In economics, these problems are known as *assignment problems*. Most previous research focused on designing mechanisms that satisfy some desired properties, including Pareto-optimality and strategy-proofness, for assignment problems with different constraints on agents' preferences [Pápai 2000b,a, 2001; Hatfield 2009; Klaus and Miyagawa 2002; Ehlers and Klaus 2003; Svensson 1999]. Many of these papers characterize *serial dictatorships* or their variants. A mechanism is a serial dictator-

[1]For example, wraps with different meats are served. Even for wraps with the same kind of meat, we still assume that an agent will have (slightly) different preferences.

ship if the agents pick their top-ranked available bundles in turn, according to a given order.

In artificial intelligence, this type of problems are known as *multiagent resource allocation (MARA)* [Chevaleyre et al. 2006]. Most previous research focused on designing *centralized* and *decentralized* (usually negotiation schemes) mechanisms with an emphasis on computational aspects of preference representation, communication, and computing the optimal assignment.

In this paper, we focus on designing mechanisms to assign indivisible items that are categorized, and each agent is required to get exactly one item from each category. Following the convention in economics, we call such problems *assignment problems with indivisible and categorized items*. Formally, suppose there are $n \geq 2$ agent and $p \geq 1$ categories of items, denoted by $\mathcal{I} = \{D_1, \ldots, D_p\}$, where each category D_i is a set of n indivisible items, that is, $|D_1| = |D_2| = \cdots = |D_p| = n$. Agents' preferences are represented by linear orders over $\mathcal{D} = D_1 \times \cdots \times D_p$. We want to design a mechanism f to assign these np items to the agent such that for every $i \leq p$, every agent gets exactly one item in D_i.

1.1. Our contributions

We first give a characterization of serial dictatorships by showing that a mechanism that allocates indivisible and categorized items and satisfies *strategy-proofness*, *non-bossiness*, and *category-wise neutrality* if and only if it is a serial dictatorship. Moreover, we also show that the three properties are a minimal set of properties that characterizes serial dictatorships.

Then, we move on to analyze a natural extension of serial dictatorships called *sequential allocation mechanisms*, which allocates the items in np stages according to a given order over all (agent,category) pairs in the following way: in stage $i = 1, \ldots, np$, suppose (a, c) is ranked at the ith place in the order, then agent a picks an item in category c. We consider two types of agents: in each stage, an *optimistic* agent always chooses the item in her top-ranked bundle that is still available, while a pessimistic agents always chooses the item that is best in the worst-case scenario. Our main results are: When each agent is either optimistic or pessimistic, for any sequential allocation mechanism, we give a lower bound on the rank of the allocated bundle for each agent (where the worst-case is taken over all agents' preference profile). Moreover, we show that all these lower bounds are tight and can be reached in the same preference profile, and more surprisingly, in this preference profile, there is a way to assign the items so that $n - 1$ agents get their top-ranked bundle, and the remaining agent gets her top-ranked or second-ranked bundle.

Most previous work in economics focused on assignment problems with different constraints on the agents' preferences, e.g. quota constraints [Pápai 2000b], monotonic preferences [Ehlers and Klaus 2003; Hatfield 2009], and additive preferences [Hatfield 2009]. Most previous work in multiagent systems focused on various compact representations, including k-additive preferences [Chevaleyre et al. 2008], CI-nets [Bouveret et al. 2009], and SCI-nets [Bouveret et al. 2010]. Recently welfare properties and computational complexity of equilibrium under serial dictatorships have been studied for additive preference [Bouveret and Lang 2011; Kalinowski et al. 2013a,b]. Our work is also related to preference representation and aggregation in multi-issue domains, especially combinatorial voting [Lang and Xia 2009].

We are not aware of any previous work that focuses on the assignment problem with categorized items (or equivalently, agents' preferences are represented by linear orders over bundles). We do not see any previous results characterization results for serial dictatorships imply ours. We are also not aware of similar results as ours for sequential allocations. The closest might be [Bouveret and Lang 2011; Kalinowski et al. 2013a] but these papers focused on the social welfare under specific utility functions, while our results reveals the worst-case rank of the allocated bundle for each single agent.

2. PRELIMINARIES

Suppose there are $n \geq 2$ agent and $p \geq 1$ categories of items, denoted by $\mathcal{I} = \{D_1, \ldots, D_p\}$, where each category D_i is a set of n indivisible items $\{1_i, \ldots, n_i\}$. Sometimes the subscripts are omitted. Agents' preferences are represented by linear orders over $\mathcal{D} = D_1 \times \cdots \times D_p$. Each element in $\mathcal{D}$ is called a *bundle*. For any $d, j \leq n$ and any $i \leq p$, we use $([j]_i, [d]_{-i})$ denote the bundle where the ith component is j and the other components are all d, where "$[\cdots]$" is used to distinguish $[j]_i$ from the variable j_i. For any $j \leq n$, let R_j denote a linear order over $\mathcal{D}$, and let $P = (R_1, \ldots, R_n)$ denote a *preference profile*. An *assignment* (or *allocation*) A is a mapping from $\{1, \ldots, n\}$ to $\mathcal{D}$ such that for every $j \leq n$, $A(j)$ is the bundle assigned to agent j, which means that for any $i \leq p$, we have $\cup_{j=1}^{n}(A(j))_i = D_i$, where $(A(j))_i$ is the item in category i that is assigned to agent j. A *mechanism* f is a mapping that takes any preference profile as input, and outputs an assignment. In this paper we sometimes use $f^j(P)$ to denote $(f(P))(j)$, that is, the bundle allocated to agent j when the preference profile is R.

We say a mechanism f satisfies *strategy-proofness*, if no agent can benefit from misreporting here preferences. f satisfies *non-bossiness*, if no agent is *bossy* in f. An agent is bossy if she can report differently to change the bundles allocated to some other agents, while keeping her own allocation unchanged. That is, j is bossy if there exists a profile P and R'_j such that $f^j(P) = f^j(P_{-j}, R'_j)$ and there exists l with $f^l(P) \neq f^l(P_{-j}, R'_j)$. f satisfies *category-wise neutrality*, if we apply any permutations to the preference profile such that each of these permutations only permutes the names of items within the same category, then the final allocation is permuted in the same way.

A mechanism is a *serial dictatorship*, if there exists a linear order $\mathcal{K}$ over $\{1, \ldots, n\}$ such that for any preference profile P, agents pick their top-ranked available bundle sequentially according to $\mathcal{K}$.

3. A CHARACTERIZATION OF SERIAL DICTATORSHIPS

In this section we characterize serial dictatorship by a minimal set of 3 properties: strategy-proofness, non-bossiness, and category-wise neutrality. Due to the space constraints, many proofs are omitted. All omitted proofs can be found on the author's homepage.

We will frequently use the following two classical lemmas for strategy-proof mechanisms. Similar ones were proved for other situations.

Lemma 1 *For any n, m, let f be a strategy-proof and non-bossy allocation mechanism. For any preference profile P and any preference profile P' such that for all $j \leq n$ and all bundles $\vec{d}$, $[f^j(P) \succ_{R_j} \vec{d}] \Longrightarrow [f^j(P) \succ_{R'_j} \vec{d}]$, we have $f(P') = f(P)$.*

PROOF. We first prove the lemma for the special case where P and P' only differ on one agent's preferences. Let j be an agent such that $R'_j \neq R_j$ and for all $\vec{d}$, $\vec{d}$, $[f^j(P) \succ_{R_j} \vec{d}] \Longrightarrow [f^j(P) \succ_{R'_j} \vec{d}]$. We will prove that $f^j(R'_j, R_{-j}) = f^j(R_j, R_{-j})$.

Suppose for the sake of contradiction $f^j(R'_j, R_{-j}) \neq f^j(R_j, R_{-j})$. If $f^j(R'_j, R_{-j}) \succ_{R_j} f^j(R_j, R_{-j})$ then it means that f is not strategy-proof since j has incentive to report R'_j when her true preferences are R_j. If $f^j(R_j, R_{-j}) \succ_{R_j} f^j(R'_j, R_{-j})$ then $f^j(R_j, R_{-j}) \succ_{R'_j} f^j(R'_j, R_{-j})$, which means that when agent j's preferences are R'_j she has incentive to report R_j, which again contradicts the assumption that f is strategy-proof. Therefore $f^j(R_j, R_{-j}) = f^j(R'_j, R_{-j})$.

By non-bossiness $f(R_j, R_{-j}) = f(R'_j, R_{-j})$. Then we recursively apply this for $j = 1, \ldots, n$, which leads to the lemma.

For any ranking R over $\mathcal{D}$ and any bundle $\vec{d} \in \mathcal{D}$, we say ranking R' is a *pushup* of $\vec{d}$ from R if R' can be obtained from R by raising the position of $\vec{d}$ while keeping the relative positions of other bundles unchanged.

Lemma 2 *Let f be a stragyproof and non-bossy mechanism. For any profile P, any $j \leq n$, any bundle $\vec{d}$, and any R'_j that is a pushup of $\vec{d}$ from R_j, either 1) $f(R'_j, R_{-j}) = f(R)$ or 2) $f(R'_j, R_{-j}) = \vec{d}$.*

PROOF. Suppose on the contrary that $f(R'_j, R_{-j})$ is neither $f(R)$ nor $\vec{d}$. If $f(R'_j, R_{-j}) \succ_R f(R)$, then f is not strategy-proof since when agent j's true preferences are R_j and other agents' preferences are R_{-j}, she has incentive to report R'_j to make her allocation better. If $f(R) \succ_R f(R'_j, R_{-j})$, then since $\vec{d} \neq f(R'_j, R_{-j})$, we have $f(R) \succ_{R'} f(R'_j, R_{-j})$. In this case when agent j's true preferences are R'_j and other agents' preferences are R_{-j}, she has incentive to report R_j to make her allocation better, which means that f is not strategy-proof.

We prove that strategy-proofness, non-bossiness, and category-wise neutrality imply Pareto-optimality.

Proposition 1 *For any m and n, any strategy-proof, non-bossy, and category-wise neutral allocation mechanism is Pareto optimal.*

PROOF. We prove the proposition by contradiction. Suppose f is a strategy-proofness, non-bossiness, category-wise neutral allocation mechanism and f is not Pareto-optimal. Let P denote a profile such that $f(P)$ is Pareto dominated by an allocation A. For any $i \leq m$, let M_i denote the permutation over D_i so that for every $j \leq n$, $(f^j(P))_i$ is permuted to $(A^j)_i$. Since bother $f(P)$ and A allocates all items, M_i is well defined. Let $M = (M_1, \ldots, M_m)$. It follows that for all $j \leq n$, $M(f^j(P)) = A^j$.

Let R'_j denote any ranking obtained from R_j by raising A^j to the top place, and then raising $f^j(P)$ to the second place if it is different from A^j, and the other bundles are ranked arbitrarily. Let R^*_j denote any ranking obtained from R_j by raising $f^j(P)$ to the top place, and then raising A^j to the second place if it is different from $f^j(P)$, and the other bundles are ranked arbitrarily. Let $P' = (R'_1, \ldots, R'_n)$ and $P^* = (R^*_1, \ldots, R^*_n)$. Since A Pareto dominates $f(P)$, by Lemma 1 we have $f(P') = f(P)$. Also by Lemma 1 we have $f(P^*) = f(P)$. By category-wise neutrality $f(M(P')) = M(f(P')) = A$. However, the only differences between $M(P')$ and P^* are the orderings among $\mathcal{D} \setminus \{A^j, f^j(P)\}$. By Lemma 1 $f(P) = f(P^*) = f(M(P')) = A$, which is a contradiction.

Lemma 3 *Given any m, n and any allocation mechanism f that satisfies strategy-proofness and non-bossiness. For any preference profile P, and any $j_1 \neq j_2 \leq n$, let $\vec{a} = f^{j_1}(P)$ and $\vec{b} = f^{j_2}(P)$, there is no $\vec{c} \in \{a_1, b_1\} \times \{a_2, b_2\} \times \cdots \times \{a_m, b_m\}$ such that $\vec{c} \succ_{R_{j_1}} \vec{a}$ and $\vec{c} \succ_{R_{j_2}} \vec{b}$.*

Theorem 1 *For any $m \geq 2$ and $n \geq 2$, an allocation mechanism is strategy-proof, non-bossy, and category-wise neutral if and only if it is a serial dictatorship. More-*

over, strategy-proofness, non-bossiness and category-wise neutrality are a minimal set of properties that characterize serial dictatorships.

PROOF. The proof is inspired by proofs in [Pápai 2000b, 2001; Hatfield 2009].[2] We first prove that strategy-proofness, non-bossiness and category-wise neutrality imply the mechanism is a serial dictatorship. Let R be a linear order over $\mathcal{D}$ that satisfies the following conditions.

— $(1,\ldots,1) \succ (2,\ldots,2) \succ \cdots \succ (n,\ldots,n)$.
— For any $i < n$, the bundles ranked between $(i,\ldots,i)$ and $(i+1,\ldots,i+1)$ are those satisfying the following two conditions: 1) at least one component is i, and 2) all components are in $\{i, i+1,\ldots,n\}$. Let K_i denote these bundles. That is, $K_i \subseteq \mathcal{D}$ and $K_i = \{\vec{d} : \forall l, d_l \geq i \text{ and } \exists l', d_{l'} = i\}$.
— For any i and any $\vec{d}, \vec{e} \in K_i$, if the number of i's in d is strictly larger than the number of i's in $\vec{e}$, then $\vec{d} \succ \vec{e}$.

Let $P = (R,\ldots,R)$. We will first prove the following claim.

Claim 1 *For any $l \leq n$, there exists $j_l \leq n$ such that $f^{j_l}(P) = (l,\ldots,l)$.*

PROOF. We prove the claim by induction on l. When $l = 1$. For the sake of contradiction suppose there is no j_l with $f^{j_l}(P) = (1,\ldots,1)$. The there exist a pair of agents j_1 and j_2 such that both $\vec{a} = f^{j_1}(P)$ and $\vec{b} = f^{j_2}(P)$ contain 1 in some components.

Let $\vec{c}$ be the bundle obtained from $\vec{a}$ by changing all components where $\vec{b}$ takes 1 to 1. More precisely,

$$\vec{c} = (c_1,\ldots,c_n) \text{ s.t. } c_i = \begin{cases} 1 & \text{if } a_i = 1 \text{ or } b_i = 1 \\ a_i & \text{otherwise} \end{cases}$$

It follows that in R, $\vec{c} \succ_R \vec{a}$ and $\vec{c} \succ_R \vec{b}$ since the number of 1's in $\vec{c}$ is strictly larger than the number of 1's in $\vec{a}$ or $\vec{b}$. However, this contradicts the assumption that f is strategy-proof and non-bossy by Lemma 3. Hence there exists $j_1 \leq n$ such that $f^{j_1}(P) = (1,\ldots,1)$.

Suppose the claim is true for $l \leq l'$. We next prove that there exists $j_{l'+1}$ such that $f^{j_{l'+1}}(P) = (l'+1,\ldots,l'+1)$. This follows from a similar reasoning as above for the $l = 1$ case. Formally, suppose for the sake of contradiction there does not exist such a $j_{l'+1}$. Then, there exist two agents who get $\vec{a}$ and $\vec{b}$ in $f(P)$, where both $\vec{a}$ and $\vec{b}$ contain $l'+1$ for some categories (but not the same). By the induction hypothesis, all components of $\vec{a}$ and $\vec{b}$ are at least $l'+1$. Let $\vec{c}$ be the bundle obtained from $\vec{a}$ by changing all components where $\vec{b}$ takes $l'+1$ to $l'+1$. We have $\vec{c} \succ_R \vec{a}$ and $\vec{c} \succ_R \vec{b}$, which is a contradiction by Lemma 3. Therefore, the claim holds for $l = l'+1$, which completes the proof.

We are ready to prove the theorem. Let $j_1,\ldots,j_n$ denote the agents in Claim 1. For any profile $P' = (R'_1,\ldots,R'_n)$, we define n bundles as follows. Let $\vec{d^1}$ denote the top-ranked bundle at R'_{j_1}, and for any $l \geq 2$, let $\vec{d^l}$ denote j_l's top-ranked available bundle given that $\vec{d^1},\ldots,\vec{d^{l-1}}$ are already chosen. That is, $\vec{d_l}$ is the bundle that is ranked in top of $\{\vec{d} : \forall l' < l, \vec{d} \cap \vec{d_{l'}} = \emptyset\}$ by R'_{j_l}. It follows that $\vec{d^1},\ldots,\vec{d^n}$ is an allocation. Then, for any $l \leq m$, we define a category-wise permutation M_l such that for all $j \leq n$, $M_l(j) = d^j_l$. Let $M = (M_1,\ldots,M_m)$. We have that for all $j \leq n$, $M(j,\ldots,j) = \vec{d^j}$. By

[2]However, we do not see an easy way to extend proofs in these papers to our setting.

category-wise neutrality and Claim 1, in $f(M(P))$ agent j gets $\vec{d^j}$. We note that for all $j \leq n$, if $\vec{e} \succ_{M(R'_j)} \vec{d^j}$ then $\vec{e} \succ_{R'_j} \vec{d^j}$, otherwise it is a contradiction to the selection of $\vec{d^j}$. Therefore by Lemma 1, $f(P') = f(M(P)) = M(f(P))$, which proves that f is a serial dictatorship.

Next, we show that strategy-proofness, non-bossiness, and category-wise neutrality are a minimal set of properties that characterize serial dictatorships by giving the following examples.

strategy-proofness is necessary: Consider the allocation mechanism that maximizes the social welfare w.r.t. the following utility functions. For any $i \leq m$ and $j \leq n$, the bundle that is ranked at the ith position of agent j's preferences gets $(m-i)(1+(\frac{1}{2m})^j)$ points. It is not hard to check that for any pair of different allocations, the social welfare are different. Therefore, the allocation mechanism satisfies non-bossiness since if agent j's allocation is the same when only agent j report differently, the set of items allocated to other agents is also the same, which implies that the optimum allocation is the same. Since the definition of utilities only depends on the positions (but not on the names of the items), the allocation mechanism satisfies category-wise neutrality. It is not hard to verify that this mechanism is not a serial dictatorship. For example, consider the case of $m = n = 2$ with $R'_1 = [11 \succ 12 \succ 22 \succ 21]$ and $R'_2 = [12 \succ 21 \succ 11 \succ 22]$. A serial dictatorship will either give 11 to agent 1 and give 22 to agent 2, or give 21 to agent 1 and give 12 to agent 2, but the allocation that maximizes social welfare is to give 12 to agent 1 and give 21 to agent 2.[3]

non-bossiness is necessary: Consider the allocation mechanism that is a serial dictatorship where agent 1 picks first, and the order over agents $\{2,\ldots,n\}$ depends on the first agents' preferences in the following way: if first component of agent 1's second choice is the same as the first component of her top choice, then the order over the rest of agents is $2 \rhd 3 \rhd \cdots \rhd n$; otherwise it is $n \rhd n-2 \rhd \cdots \rhd 2$. It is not hard to verify that this mechanism satisfies strategy-proofness and category-wise neutrality, and is not a serial dictatorship (where the order must be fixed before seeing the preference profile).

category-wise neutrality is necessary: Consider the allocation mechanism that is a serial dictatorship where agent 1 picks first, and the order over agents $\{2,\ldots,n\}$ depends on the allocation of agent 1 in the following way: if agent 1 gets $(1,\ldots,1)$, then the order over the rest of agents is $2 \rhd 3 \rhd \cdots \rhd n$; otherwise it is $n \rhd n-2 \rhd \cdots \rhd 2$. It is not hard to verify that this mechanism satisfies strategy-proofness and non-bossiness, and is not a serial dictatorship (where the order must be fixed before seeing the preference profile).

4. SEQUENTIAL ALLOCATION MECHANISMS

A sequential allocation is defined to be a total order $\mathcal{O}$ over $\{1,\ldots,n\} \times \{1,\ldots,p\}$, where (j,i) means that it is agent j's turn to choose an item from D_i. Let $\mathcal{O}^{-1}(j,i) \in [1,np]$ denote the position of (j,i) in $\mathcal{O}$.

Given $\mathcal{O}$, the sequential allocation is a distributed protocol that works in np steps as follows. For each step t, suppose the t-th element in $\mathcal{O}$ is (j,i), that is, $t = \mathcal{O}^{-1}(j,i)$. Then, agent j is required to choose an item from D_i that is still available (meaning that no agent has chosen that value before). Then, agent j's choice is sent to the center and broadcast among all agents. The protocol is illustrated in Algorithm 1.

As for combinatorial auctions and combinatorial voting, the advantage of Algorithm 1 are two-fold.

[3]The utility functions are only used to avoid ties in allocations. Any utility functions where there are no ties satisfy non-bossiness and category-wise neutrality, but some of them are serial dictatorships.

Algorithm 1: Sequential allocation protocol.

Input: An order $\mathcal{O}$ over $\{1, \ldots, n\} \times \{1, \ldots, p\}$.
1 Broadcast $\mathcal{O}$ to all agents.
2 **for** $t = 1$ *to* np **do**
3 Let (j, i) be the t-th in $\mathcal{O}$.
4 Ask agent j to choose an available item $d_{j,i}$ from D_i.
5 Broadcast $d_{j,i}$ to all agents.
6 **end**

First, it has a low communication cost compared to the straightforward centralized system, where reporting a full ranking over $\mathcal{D}$ requires $O(n^p p \log n)$ bits for each agent, so totally $O(n^{p+1} p \log n)$. For sequential allocation, broadcasting $\mathcal{O}$ (Step 1) uses $O(n(np \log np)) = O(n^2 p \log np)$ bits, then in each round communicate the allocation from the active agent, and then broadcast it to the other agents takes $O(n \log n)$. Since there are totally np rounds, so the total communication complexity for Algorithm 1 is $O(n^2 p \log n + n(np \log np + p \log p) = O(n^2 p \log np)$.

Second, agents are more comfortable choosing a value for an category per step compared to reporting a full ranking over the n^p bundles in $\mathcal{D}$, when p is not too small.

Therefore, sequential allocation is an *indirect mechanism*, since the message space for the agents are different from their preference space. This is the main reason why we can have significant communicational savings. For indirect mechanisms, it is hard (if not impossible) to define the behavior of a truthful agent.

In this paper, we will investigate two types of "truthful" agents under sequential allocation mechanisms.

— Setting 1: *optimistic* agents. When an optimistic agent j is active and is asked to report a value from D_i, she will choose the ith component of the top-ranked bundle that is still available, given the results of allocation in previous steps.
— Setting 2: *pessimistic* agents. When a pessimistic agent j is active and is asked to report a value from D_i, she will choose a value $d_{j,i}$ so that the worst-case available bundle is optimized given the results of previous rounds and $d_{j,i}$.

Example 1 *Let $n = 3$, $p = 2$. Three agents' preferences are as follows, where 12 represents $(1, 2)$ and for each agent, "others" represents any order over the bundles not specified in the context (and it will not affect the outcome of sequential allocation in this example).*

Agent 1 (optimistic): $12 \succ 21 \succ$ others $\succ 11$
Agent 2 (optimistic): $32 \succ$ others $\succ 22$
Agent 3 (pessimistic): $13 \succ$ others $\succ 33 \succ 31 \succ 23$

Let $\mathcal{O} = [(1, 1) \rhd (2, 2) \rhd (3, 1) \rhd (3, 2) \rhd (2, 1) \rhd (1, 2)]$. Suppose agent 1 and agent 2 are optimistic and agent 3 is pessimistic.

When $t = 1$, $(1, 2)$ is the top-ranked available item for agent 1. Since agent 1 is optimistic, she chooses 1. When $t = 2$, $(3, 2)$ is the top-ranked available item for agent 2. Since agent 2 is optimistic, she chooses 2. When $t = 3$, the available bundles are $\{2, 3\} \times \{1, 3\}$. If agent 3 chooses 2, then the worst-case available bundle is $(2, 3)$, and if agent 3 chooses 3 then the worst-case available bundle is $(3, 1)$. Since agent 3 prefers $(3, 1)$ to $(2, 3)$, she will choose 3 from D_1. When $t = 4$, the available bundles are $\{3\} \times \{1, 3\}$, and agent 3 will choose 3 from D_2. Then when $t = 5$, agent 2 choses 2_1 and when $t = 6$, agent 1 choses 1_2. The final allocation is: agent 1 gets $(1, 1)$, agent 2 gets $(2, 2)$, and agent 3 gets $(3, 3)$.

5. EFFICIENCY OF SEQUENTIAL ALLOCATION

In this section, we assume that each agent is either optimistic or pessimistic. We note that this is much more general than assuming that the agents are either all optimistic or all pessimistic. Given any sequential allocation $\mathcal{O}$, we will characterize the worst-case rank of the allocation by $\mathcal{O}$ for all agents. To do so, given $\mathcal{O}$, we define the following notation for any j.

— Let $\mathcal{O}_j$ denote the order over $\{1, \ldots, p\}$ according to which agent j choose values for categories in $\mathcal{I}$.
— Let K_j denote the smallest natural number such that no agent can "interrupt" agent j from choosing her top-ranked bundle that is still available for the categories $\mathcal{O}_j(K_j), \mathcal{O}_j(K_j+1), \ldots, \mathcal{O}_j(p)$. Formally, K_j is the smallest number so that for any i' with $K_j < i' \leq p$, between the round when agent j chooses $\mathcal{O}_j(K_j)$ and the round when agent j chooses $\mathcal{O}_j(i')$ according to $\mathcal{O}$ in Algorithm 1, no agent chooses a value for $\mathcal{O}_j(i')$. Note that K_j is defined solely by $\mathcal{O}$, which means that it does not depend on the agents' preferences (which will be considered in the worst-case analysis). Also note that the K_j represents a position in $\mathcal{O}_j$, which is not necessarily the K_jth category.
— For any $i \leq p$, let $k_{j,i}$ denote the number of values of D_i that have *not* been chosen by other agents before agent j chooses a value from D_i. Formally, $k_{j,i} = 1 + |\{(j', i) : (j, i) \rhd_{\mathcal{O}} (j', i)\}|$. Equivalently $k_{j,i} = n - |\{(j', i) : (j', i) \rhd_{\mathcal{O}} (j, i)\}|$.

Example 2 *Let* $\mathcal{O} = [(1,1) \rhd (1,2) \rhd (1,3) \rhd (2,1) \rhd (2,2) \rhd (2,3) \rhd (3,1) \rhd (3,2) \rhd (3,3)]$. *That is, Algorithm 1 with* $\mathcal{O}$ *is a serial dictatorship. Then* $\mathcal{O}_1 = \mathcal{O}_2 = \mathcal{O}_3 = 1 \rhd 2$. $K_1 = K_2 = K_3 = 1$. $k_{1,1} = k_{1,2} = 3$, $k_{2,1} = k_{2,2} = 2$, $k_{3,1} = k_{3,2} = 1$.

Let $\mathcal{O}$ *be the order in Example 1, that is,* $\mathcal{O} = [(1,1) \rhd (2,2) \rhd (3,1) \rhd (3,2) \rhd (2,1) \rhd (1,2)]$. $\mathcal{O}_1 = 1 \rhd 2$. $K_1 = 2$ *since* $(2,2)$ *is between* $(1,1)$ *and* $(1,2)$ *in* $\mathcal{O}$. $k_{1,1} = 3$, $k_{1,2} = 1$.
$\mathcal{O}_2 = 2 \rhd 1$. $K_2 = 2$ *since* $(3,1)$ *is between* $(2,2)$ *and* $(2,1)$. $k_{2,1} = 1$, $k_{2,2} = 3$.
$\mathcal{O}_3 = 1 \rhd 2$. $K_3 = 1$ *since between* $(3,1)$ *and* $(3,2)$ *in* $\mathcal{O}$, *no agent chooses an item from* D_2. $k_{3,1} = 2$, $k_{3,2} = 2$.

Proposition 2 *For any* $j \leq n$, *we have the following bounds.*

— **Lower bounds for optimistic agents:** *the bundle assigned to an optimistic agent* j *is ranked no lower than the* $\prod_{l=K_j}^{p} k_{j,O_j(l)}$*-th position from the bottom.*
— **Lower bounds for pessimistic agents:** *the bundle assigned to a pessimistic agent* j *is ranked no lower than the* $(1 + \sum_{l=1}^{p}(k_{j,O_j(l)} - 1))$*-th position from the bottom.*

We note that in Proposition 2, K_j's are only used to define lower bounds for optimistic agents, and $k_{j,\mathcal{O}_j(i)}$ for all $i < K_j$ are only used to define lower bounds if agent j is pessimistic.

Our main theorem in this section states that all these lower bounds can be achieved in one preference profile. Moreover, for the same profile there exists an allocation where almost all agents get their top-ranked bundle (and the only person who may not get her top-ranked bundle gets her second-ranked bundle).

Theorem 2 *For any sequential allocation* $\mathcal{O}$, *and each agent is fixed to be optimistic or pessimistic, there exists a preference profile* P *such that:*

(1) *an optimistic agent* j*'s allocation is ranked at the* $\prod_{l=K_j}^{p} k_{j,O_j(l)}$*-th position from the bottom;*
(2) *a pessimistic agent* j*'s allocation is ranked at the* $(1 + \sum_{l=1}^{p}(k_{j,O_j(l)} - 1))$*-th position from the bottom;*

(3) *there exists an allocation where at least* $n-1$ *agents get their top-ranked bundles, and the remaining agent is allocated her top-ranked or second-ranked bundle. Moreover, if the first agent in* $\mathcal{O}$ *is pessimistic, then there exists an allocation where all agents get their top-ranked bundle.*

We emphasize that in the theorem, whether an agent is optimistic or pessimistic is fixed before we construct the preference profile.

Example 3 *Let* $\mathcal{O}$ *be the order in Example 1 and suppose agent* 1 *and* 2 *are optimistic while agent* 3 *is pessimistic.*

By Theorem 2 and Example 2, there exists a profile P *such that after applying the sequential allocation* $\mathcal{O}$*, the bundle agent* 1 *receives is ranked at* $k_{1,2} = 1$*the last position; the bundle agent* 2 *receives is ranked at the last position; and the bundle agent* 3 *receives is ranked at the* 3*rd position from the bundle. Moreover, there exists an allocation agent* 2 *and* 3 *get their top-ranked bundles and agent* 1 *gets her second-ranked bundle. In fact, the preference profile described in Example 1 satisfies all these conditions.*

6. FUTURE WORK

We feel that the assignment problems with categorized items provide a framework to apply many techniques developed in other fields of combinatorial preference representation and aggregation. In the future we plan to extend our results to other assignment problems with category constraints, for example the assignment problems where agents have indifference preferences, or some category contains more/less than n items, and the agents can get more/less than 1 item from each category. We have already mentioned that the structure of category constraints allow us to explore other preference representation languages that were not suitable for non-categorized domains, especially CP-nets [Boutilier et al. 2004], LP-trees [Booth et al. 2010], and soft constraints [Pozza et al. 2011]. We also plan to analyze the behavior and effect of strategic agents or regret-minimizing agents. We can also study probabilistic assignment mechanisms.

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Appendix:Proofs

Lemma 3

PROOF. Suppose for the sake of contradiction such a $\vec{c}$ exist. Let $\vec{d}$ be the allocation such that $\vec{c} \cup \vec{d} = \vec{a} \cup \vec{b}$. More precisely, for all $i \leq m$, $\{c_i, d_i\} = \{a_i, b_i\}$. For example, if $\vec{a} = 1213$ and $\vec{b} = 2431$, then one possibility is $\vec{c} = 1211$.

The rest of the proof derives a contradiction by proving the following observations in sequence illustrated in Table I, where in each step, we will prove that the boxed bundles are the winners for agent j_1 and agent j_2, and all other agents get their top-ranked bundles.

Table I. Proof sketch of Lemma 3.

Step 1	Step 2	Step 3
$\hat{R}_{j_1} : \vec{c} \succ \boxed{\vec{a}} \succ \vec{d} \succ \vec{b} \succ$ others	$\hat{R}_{j_1} : \vec{c} \succ \boxed{\vec{a}} \succ \vec{d} \succ \vec{b} \succ$ others	$\bar{R}_{j_1} : \vec{c} \succ \boxed{\vec{b}} \succ \vec{a} \succ \vec{d} \succ$ others
$\hat{R}_{j_2} : \vec{c} \succ \boxed{\vec{b}} \succ \vec{a} \succ \vec{d} \succ$ others	$\bar{R}_{j_2} : \vec{c} \succ \vec{a} \succ \boxed{\vec{b}} \succ \vec{d} \succ$ others	$\bar{R}_{j_2} : \vec{c} \succ \boxed{\vec{a}} \succ \vec{b} \succ \vec{d} \succ$ others
Other $j : f^j(P) \succ$ others	Other $j : f^j(P) \succ$ others	Other $j : f^j(P) \succ$ others

Step 4	Step 5	Step 6
$\bar{R}_{j_1} : \vec{c} \succ \boxed{\vec{b}} \succ \vec{a} \succ \vec{d} \succ$ others	$\bar{\bar{R}}_{j_1} : \vec{c} \succ \vec{a} \succ \boxed{\vec{b}} \succ \vec{d} \succ$ others	$\bar{\bar{R}}_{j_1} : \vec{c} \succ \boxed{\vec{a}} \succ \vec{b} \succ \vec{d} \succ$ others
$\bar{\bar{R}}_{j_2} : \vec{c} \succ \boxed{\vec{a}} \succ \vec{d} \succ \vec{b} \succ$ others	$\bar{\bar{R}}_{j_2} : \vec{c} \succ \boxed{\vec{a}} \succ \vec{d} \succ \vec{b} \succ$ others	$\bar{R}_{j_2} : \vec{c} \succ \vec{a} \succ \boxed{\vec{b}} \succ \vec{d} \succ$ others
Other $j : f^j(P) \succ$ others	Other $j : f^j(P) \succ$ others	Other $j : f^j(P) \succ$ others

Step 1. Let $\hat{R}_{j_1} = [\vec{c} \succ \vec{a} \succ \vec{d} \succ \vec{b} \succ \text{others}]$, $\hat{R}_{j_2} = [\vec{c} \succ \vec{b} \succ \vec{a} \succ \vec{d} \succ \text{others}]$, where "others" represents an arbitrary linear order over the remaining bundles, and for any $j \neq j_1, j_2$, let $\hat{R}_j = [f^j(P) \succ \text{others}]$. By Lemma 1, $f(\hat{P}) = f(P)$.
Step 2. Let $\bar{R}_{j_2} = [\vec{c} \succ \vec{a} \succ \vec{b} \succ \vec{d} \succ \text{others}]$ be a pushup of $\vec{a}$ from $\hat{R}_{j_2}$. We will show that $f(\bar{R}_{j_2}, \hat{R}_{-j_2}) = f(\hat{P}) = f(P)$. Since $\bar{R}_{j_2}$ is a pushup of $\vec{a}$ from $\hat{R}_{j_2}$, by Lemma 2, $f^{j_2}(\bar{R}_{j_2}, \hat{R}_{-j_2})$ is either $\vec{a}$ or $\vec{b}$. We will show that the former case is impossible. Suppose for the sake of contradiction $f^{j_2}(\bar{R}_{j_2}, \hat{R}_{-j_2}) = \vec{a}$, then $f^{j_1}(\bar{R}_{j_2}, \hat{R}_{-j_2})$ cannot be $\vec{c}$, $\vec{a}$, or $\vec{d}$ since otherwise some item will be allocated twice. This means that j_1 prefers $\vec{d}$ to $f^{j_1}(\bar{R}_{j_2}, \hat{R}_{-j_2})$. It follows that $f(\bar{R}_{j_2}, \hat{R}_{-j_2})$ is Pareto dominated by the allocation where j_1 gets $\vec{d}$, j_2 gets $\vec{c}$, and any other agent j gets $f^j(P)$. This contradicts Pareto-optimality of f (Proposition 1). Hence $f^{j_2}(\bar{R}_{j_2}, \hat{R}_{-j_2}) = \vec{b} = f^{j_2}(\hat{P})$. By non-bossiness we have $f(\bar{R}_{j_2}, \hat{R}_{-j_2}) = f(\hat{P}) = f(P)$.
Step 3. Let $\bar{R}_{j_1} = [\vec{c} \succ \vec{b} \succ \vec{a} \succ \vec{d} \succ \text{others}]$ be a pushup of $\vec{b}$ from $\hat{R}_{j_1}$. We will show that in $f(\bar{R}_{j_1}, \bar{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}})$, j_1 gets $\vec{b}$, j_2 gets $\vec{a}$, and all other agents get the same items as in $f(P)$. Since $\bar{R}_{j_1}$ is a pushup of $\vec{b}$ from $\hat{R}_{j_1}$, by Lemma 2, $f^{j_1}(\bar{R}_{j_1}, \bar{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}})$ is either $\vec{a}$ or $\vec{b}$. We now show that the former case is impossible by contradiction. Suppose for the sake of contradiction that $f^{j_1}(\bar{R}_{j_1}, \bar{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}}) = \vec{a}$. Then by non-bossiness $f^{j_2}(\bar{R}_{j_1}, \bar{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}}) = \vec{b}$. This means that $f(\bar{R}_{j_1}, \bar{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}})$ is Pareto-dominated by the allocation where j_1 gets $\vec{b}$, j_2 gets $\vec{a}$, and all other agents get the same items as in $f(P)$. Again, this violates Pareto-optimality.
Step 4. Let $\bar{\bar{R}}_{j_2} = [\vec{c} \succ \vec{a} \succ \vec{d} \succ \vec{b} \succ \text{others}]$ be a pushup of $\vec{d}$ from $\bar{R}_{j_2}$. By Lemma 1, $f(\bar{R}_{j_1}, \bar{\bar{R}}_{j_2}, \hat{R}_{-\{j_1,j_2\}}) = f(\bar{R}_{j_1}, \bar{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}})$.
Step 5. Let $\bar{\bar{R}}_{j_1} = [\vec{c} \succ \vec{a} \succ \vec{b} \succ \vec{d} \succ \text{others}]$ be a pushup of $\vec{b}$ from $\bar{R}_{j_1}$. We will show that $f(\bar{\bar{R}}_{j_1}, \bar{\bar{R}}_{j_2}, \hat{R}_{-\{j_1,j_2\}}) = f(\bar{R}_{j_1}, \bar{\bar{R}}_{j_2}, \hat{R}_{-\{j_1,j_2\}})$. Since $\bar{\bar{R}}_{j_1}$ is a pushup of $\vec{a}$ from $\bar{R}_{j_1}$, by Lemma 2, $f^{j_1}(\bar{\bar{R}}_{j_1}, \bar{\bar{R}}_{j_2}, \hat{R}_{-\{j_1,j_2\}})$ is either $\vec{a}$ or $\vec{b}$. We now show that the former case is impossible by contradiction. Suppose for the sake of contradiction that $f^{j_1}(\bar{\bar{R}}_{j_1}, \bar{\bar{R}}_{j_2}, \hat{R}_{-\{j_1,j_2\}}) = \vec{a}$. Then in $f(\bar{\bar{R}}_{j_1}, \bar{\bar{R}}_{j_2}, \hat{R}_{-\{j_1,j_2\}})$, agent j_2 cannot get $\vec{c}$, $\vec{a}$, or $\vec{d}$, which means that $f(\bar{\bar{R}}_{j_1}, \bar{\bar{R}}_{j_2}, \hat{R}_{-\{j_1,j_2\}})$ is Pareto-dominated by the allocation where j_1 gets $\vec{c}$, j_2 gets $\vec{d}$, and all other agents get the same items as in $f(P)$. This violates Pareto-optimality. So $f^{j_1}(\bar{\bar{R}}_{j_1}, \bar{\bar{R}}_{j_2}, \hat{R}_{-\{j_1,j_2\}}) = \vec{b}$. By non-bossiness $f(\bar{\bar{R}}_{j_1}, \bar{\bar{R}}_{j_2}, \hat{R}_{-\{j_1,j_2\}}) = f(\bar{R}_{j_1}, \bar{\bar{R}}_{j_2}, \hat{R}_{-\{j_1,j_2\}})$.
Step 6. We note that $\bar{\bar{R}}_{j_1}$ is a pushup of $\vec{b}$ from $\hat{R}_{j_1}$ (and $\vec{b}$ is still below $\vec{a}$). By Lemma 1, $f(\bar{\bar{R}}_{j_1}, \bar{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}}) = f(\hat{R}_{j_1}, \bar{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}})$. We note that the right hand side is the profile in Step 3.
Contradiction. Finally, the observations in Step 5 and Step 6 imply that when the true preferences are as in Step 6, agent j_2 has incentive to report $\bar{\bar{R}}_{j_2}$ in Step 5 to make her allocation better (from $\vec{b}$ to $\vec{a}$). This contradicts the strategy-proofness of f and completes the proof.

Proposition 2

PROOF. We first prove the proposition for an optimistic agent j, w.l.o.g. let $\mathcal{O}_j = 1 \triangleright 2 \triangleright \cdots \triangleright p$. That is, agent j chooses items from categories $1, \ldots, p$ in sequence in $\mathcal{O}$. Then in the beginning of round $t = \mathcal{O}^{-1}(j, K_j)$ in Algorithm 1, agent j has already chosen items from $D_1, \ldots, D_{K_j-1}$, and is ready to choose an item from D_{K_j}. For any l

with $K_j \leq l \leq p$, we let $D_{l,t}$ denote the set of remaining items in D_l, that is, the items that are not chosen by another agent in previous rounds. We have $|D_{l,t}| = k_{j,l}$. Let $(d_{j,1}, \ldots, d_{j,K_j-1}) \in D_1 \times \cdots \times D_{K_j-1}$ denote the items agent j has chosen in previous rounds. It follows that in round $t = \mathcal{O}^{-1}(j, K_j)$ the following $\prod_{l=K_j}^{p} k_{j,\mathcal{O}_j(l)}$ bundles are available for agent j:

$$\mathcal{D}_{\mathcal{O}^{-1}(j,K_j)} = (d_{j,1}, \ldots, d_{j,K_j-1}) \times \prod_{l=K_j}^{p} D_{l,t}$$

We will show that an optimistic agent j is guaranteed to obtain her top-ranked bundle in this set $\mathcal{D}_{\mathcal{O}^{-1}(j,K_j)}$. Intuitively this holds because by definition of K_j, for any $l \geq K_j$, when it is agent j's round to choose an item from D_l, the l-th component of her top-ranked bundle in $\mathcal{D}_{\mathcal{O}^{-1}(j,K_j)}$ is always available.

Formally, let $\vec{d_j} = (d_{j,1}, \ldots, d_{j,p})$ denote agent j's top-ranked bundle in $\mathcal{D}_{\mathcal{O}^{-1}(j,K_j)}$. We prove that agent j will choose $d_{j,l}$ from D_l in round $\mathcal{O}^{-1}(j, l)$ by induction on l. The base case $l = K_j$ is straightforward, since agent j is optimistic. Suppose she has chosen $d_{K_j}, d_{K_j+1}, \ldots, d_{l'}$ for some $l' \geq K_j$. Then in round $\mathcal{O}^{-1}(j, l'+1)$ when agent j is about to choose an item from $D_{l'+1}$, the following bundles are available:

$$\mathcal{D}_{\mathcal{O}^{-1}(j,l'+1)} = (d_{j,1}, \ldots, d_{j,l'}) \times \prod_{l=l'+1}^{p} D_{l,t^*}$$

This is because by the induction hypothesis, $(d_{j,1}, \ldots, d_{j,l'})$ are chosen previously. Then, by the definition of K_j, for any $l \geq l'+1$ no agent chose an item from D_l between round $\mathcal{O}^{-1}(j, K_j)$ and round $\mathcal{O}^{-1}(j, l')$. Hence the remaining items in D_l is still the same as that in round $\mathcal{O}^{-1}(j, K_j)$. This means that $\vec{d_j} \in \mathcal{D}_{\mathcal{O}^{-1}(j,l'+1)}$. Because $\mathcal{D}_{\mathcal{O}^{-1}(j,l'+1)} \subseteq \mathcal{D}_{\mathcal{O}^{-1}(j,K_j)}$, $\vec{d_j}$ is still agent j's top-ranked available bundle. Hence agent j will choose $d_{j,l'+1}$. This proves that the claim is true for $l = l'+1$, which means that it holds for all $l \leq p$. Therefore, agent j gets $\vec{d_j}$, which is ranked in the top of $\mathcal{D}_{\mathcal{O}^{-1}(j,K_j)}$. We note that $|\mathcal{D}_{\mathcal{O}^{-1}(j,K_j)}| = \prod_{l=K_j}^{p} k_{j,O_j(l)}$. This proves the proposition for optimistic agents.

We next prove the proposition for any pessimistic agent j. Let $\vec{d_j} = (d_{j,1}, \ldots, d_{j,p})$ denote her final allocation. Since agent j is pessimistic, for any $1 \leq l \leq p$, in round $t^* = \mathcal{O}^{-1}(j, l)$ agent j chose $d_{j,l}$ from D_{l,t^*}, we must have that for any $d'_l \in D_{l,t^*}$ with $d'_l \neq d_{j,l}$, there exists an bundle $(d_{j,1}, \ldots, d_{j,l-1}, d'_l, \ldots, d'_p)$ that is ranked below $\vec{d_j}$. The number of all such bundles is $\prod_{l=1}^{p}(k_{j,O_j(l)} - 1)$, which proves the proposition for pessimistic agents.

Theorem 2

PROOF. Given $\mathcal{O}$ and the information on whether each agent j is optimistic or pessimistic, we will construct a preference profile P such that in $\mathcal{O}(P)$, for all $j \leq n$, agent j obtains $(j, \ldots, j)$.

We prove the theorem in the following three steps: the **defining bottom bundles** step specifies a set of bundles that are ranked in the bottom positions for each agent j, and require that $(j, \ldots, j)$ is ranked in the top of them. The **defining top bundles** step specifies top-1 and sometimes also top-2 bundles for each agent. In both steps we have different ways to define these bundles for optimistic agents and pessimistic agents respectively. Finally in third step **defining the preference profile** we take any preference profile that extends the partial orders constructed in the first two steps, and then show that it satisfies all three properties in the theorem. The construction is summarized in Table II (optimistic agents) and Table III (pessimistic agents).

We first introduce some notation that will be used to define the preference profile in Step 1 and Step 2. Let $\mathcal{O}(1) = (j_1, i_1)$. That is, agent j_1 is the first to choose an item, and she chooses from category D_{i_1}. Let L_{i_1} denote the order over $\{1, \dots, n\}$ that represents the order over the *agents* to choose items from D_{i_1} in $\mathcal{O}$. That is, $j \rhd_{L_{i_1}} j'$ if and only if $(j, i_1) \rhd_{\mathcal{O}} (j', i_1)$. By definition we have $j_1 = L_{i_1}(1)$. For any $j \leq n$, we let $Pred_{i_1}(j) = L_{i_1}(L_{i_1}^{-1}(j) - 1)$ denote the predecessor of agent j in L_{i_1}, that is, the latest agent who chose from category i_1 before agent j chooses from category i_1. If $j = 1$, then we let the last agent in L_{i_1} be her predecessor, that is, $Pred_{i_1}(1) = L_{i_1}(n)$.

Step 1: defining bottom bundles. Intuitive, in order to match the lower bounds shown in the proof of Proposition 2, we must have the bundles described in the proof of Proposition 2 are the *only* bundles that are ranked below $(j, \dots, j)$ by agent j. This is the preference profile we will construct.

For all i and t, we first define $D^*_{i,t}$ to be a subset of $D_i = \{1, \dots, n\}$ such that $q \in D^*_{i,t}$ if and only if agent q has not chosen an item from D_i before the t-th round. We note that $D^*_{i,t}$ is determined by i, t, and $\mathcal{O}$, which means that it does not depend on the agents' preferences and behavior in previous rounds (later we will show that in each round (j, i) the active agent j will choose j from D_i). Formally,

$$D^*_{i,t} = \{q \leq n : \mathcal{O}^{-1}(q, i) \geq t\}$$

By definition, if $\mathcal{O}(t) = (j, i)$ then $j \in D^*_{i,t}$.

For any $1 \leq l \leq p$, we let $t^*_{j,l} = \mathcal{O}^{-1}(j, \mathcal{O}_j(l))$. That is, $t^*_{j,l}$ is the round where agent j chooses the value for the l-th category in $\mathcal{O}_j$. For each agent j we specify their bottom ranked bundles as follows.

— If agent j is optimistic, then the following bundles are ranked in the bottom of her preferences:

$$\text{BottomBundles}_j = (j_{\mathcal{O}_j(1)}, \dots, j_{\mathcal{O}_j(K_j - 1)}) \times \prod_{l=K_j}^{p} D^*_{\mathcal{O}_j(l), t^*_{j,l}}$$

where $(j, \dots, j)$ is ranked on the top of these bundles, and the order among the rest bundles is defined arbitrarily. We recall that $D^*_{\mathcal{O}_j(l), t^*_{j,l}}$ consists of the items that are available for agent j when she chooses from the l-th category in $\mathcal{O}_j$. It follows that $(j, \dots, j)$ is ranked in the $(\prod_{l=K_j}^{p} k_{j,O_j(l)})$-th position from the bottom by agent j.

— If agent j is pessimistic, then we first define the following bundles:

$$\text{BottomBundles}_j = \bigcup_{l=1}^{p} \bigcup_{d \in D^*_{\mathcal{O}_j(l), t^*_{j,l}}} \{([d]_{\mathcal{O}_j(l)}, [j]_{-\mathcal{O}_j(l)})\}$$

Bundles in BottomBundles_j are (partially) ranked as follows: first, $(j, \dots, j)$ is ranked in the top; then, for any $1 \leq l_1 < l_2 \leq p$ and any $d_1 \in D^*_{\mathcal{O}_j(l_1), t^*_{j,l_1}}$ and $d_2 \in D^*_{\mathcal{O}_j(l_2), t^*_{j,l_2}}$ with $d_1 \neq j$ and $d_2 \neq j$, we rank $([d_1]_{\mathcal{O}_j(l_1)}, [j]_{-\mathcal{O}_j(l_1)})$ below $([d_2]_{\mathcal{O}_j(l_2)}, [j]_{-\mathcal{O}_j(l_2)})$.

If $j \neq j_1$, then we simply let BottomBundles_j (with the partial orders specified above) be the bundles ranked in the bottom.

If $j = j_1$, then we move $([Pred_{i_1}(j)]_{i_1}, [j]_{-i_1})$ to the bottom place (which does not violate the partial order specified in BottomBundles_j), replace it by $(Pred_{i_1}(j), \dots, Pred_{i_1}(j))$, and then let these be the ranked in the bottom positions of agent j's preferences. That is, the bottom preferences are:

$$(j,\ldots,j) \succ (\text{BottomBundles}_j \setminus \{(j,\ldots,j),([Pred_{i_1}(j)]_{i_1},[j]_{-i_1})\}) \succ (Pred_{i_1}(j),\ldots,Pred_{i_1}(j))$$

In both cases $(j,\ldots,j)$ is ranked at the $(1+\prod_{l=K_j}^{p}(k_{j,O_j(l)}-1))$-th position from the bottom.

Step 2: defining top bundles. We now specify the top two bundles (sometimes only the top bundle) for optimistic agents, and show that they will not conflict our constructions in Step 1. For any optimistic agent j,

— if $j \neq j_1$, then there are the following two cases:
  — case 1: $K_j = 1$. We let $([Pred_{i_1}(j)]_{i_1},[j]_{-i_1})$ be the top-ranked bundle of agent j.
  — case 2: $K_j > 1$. We let $([Pred_{i_1}(j)]_{i_1},[j]_{-i_1})$ be the top-ranked bundle of agent j. Moreover, if $i_1 \neq \mathcal{O}_j(K_j)$, then we rank $([Pred_{\mathcal{O}_j(K_j)}(j)]_{\mathcal{O}_j(K_j)},[j]_{-\mathcal{O}_j(K_j)})$ at the second position. We recall that $Pred_{\mathcal{O}_j(K_j)}(j)$ is the predecessor of j in $L_{\mathcal{O}_j(K_j)}$.

  These do not conflict the preferences specified in Step 1 because item $Pred_{i_1}(j)$ in D_{i_1} is not available for agent j when she is about to choose an item in D_{i_1}, and item $Pred_{\mathcal{O}_j(K_j)}(j)$ in $D_{\mathcal{O}_j(K_j)}$ is not available for agent j when she is about to choose an item in $D_{\mathcal{O}_j(K_j)}$.

— If $j = j_1$, then there are the following two cases:
  — case 1: $K_j = 1$. Then since $(j_1,i_1) = \mathcal{O}(1)$, we have that for all i, $D^*_{i,\mathcal{O}^{-1}(j,i)} = D_i$, which means that agent j is guaranteed to be able to get her top-ranked bundle in sequential allocation. In this case we let $(j,\ldots,j)$ be agent j's top-ranked bundle and let $([L_{i_1}(n)]_{i_1},[j]_{-i_1})$ be ranked in agent j's second position. These do not conflict the preferences specified in Step 1 because in this case j is guaranteed to get her top-ranked bundle, so that Step 1 only specifies that $(j,\ldots,j)$ be ranked in the top position.
  — case 2: $K_j > 1$. We rank $([Pred_{\mathcal{O}_j(K_j)}(j)]_{\mathcal{O}_j(K_j)},[j]_{-\mathcal{O}_j(K_j)})$ at the top position. We then rank $([L_{i_1}(n)]_{i_1},[j]_{-i_1})$ at the second position. Since $i_1 = \mathcal{O}_j(1)$, we have $\mathcal{O}_j(K_j) \neq i_1$, which means that $([Pred_{\mathcal{O}_j(K_j)}(j)]_{\mathcal{O}_j(K_j)},[j]_{-\mathcal{O}_j(K_j)}) \neq ([L_{i_1}(n)]_{i_1},[j]_{-i_1})$. These do not conflict the preferences specified in Step 1 because category i_1 is agent j_1's first category in $\mathcal{O}_{j_1}$, which means that $([L_{i_1}(n)]_{i_1},[j]_{-i_1}) \notin \text{BottomBundles}_{j_1}$; also $Pred_{\mathcal{O}_j(K_j)}(j)$ is not available when agent j_1 is about to choose an item for category $\mathcal{O}_j(K_j)$, which means that $([Pred_{\mathcal{O}_j(K_j)}(j)]_{\mathcal{O}_j(K_j)},[j]_{-\mathcal{O}_j(K_j)}) \notin \text{BottomBundles}_{j_1}$.

For any pessimistic agent j, we simply let her top-ranked bundle be $([Pred_{i_1}(j)]_{i_1},[j]_{-i_1})$ (we recall that in case $j = j_1$, $Pred_{i_1}(j_1) = L_{i_1}(n)$). We claim that preferences specified in the second step do not conflict preferences specified in the first step for bottom bundles.

— If $j \neq j_1$, then we need to show that $([Pred_{i_1}(j)]_{i_1},[j]_{-i_1}) \notin \text{BottomBundles}_j$. When agent j is about to choose her item from D_{i_1}, agent $Pred_{i_1}(j)$ has already chosen her item from D_{i_1}, which means that $Pred_{i_1}(j)$ is unavailable for agent j. This means that $([Pred_{i_1}(j)]_{i_1},j_{-i_1}) \notin \text{BottomBundles}_j$.
— if $j = j_1$, then by definition (see Table III) $([Pred_{i_1}(j)]_{i_1},[j]_{-i_1})$ is replaced by $(Pred_{i_1}(j),\ldots,Pred_{i_1}(j))$ in BottomBundles_j, which means that it can be ranked in the top.

Step 3: defining the preference profile. For any j, let R_j be an arbitrary linear order over $\mathcal{D}$ that satisfies all constraints defined in the previous two steps. Let $P =$

$(R_1, \ldots, R_n)$. We now show by induction on t that if we apply the sequential allocation $\mathcal{O}$ to P, then for all $j \leq n$, agent j will get $(j, \ldots, j)$.

When $t = 1$, agent j_1 chooses an item from D_{i_1}. If j_1 is optimistic, then it is not hard to check that the i_1 component of the top-ranked bundle of R_{j_1} is j_1 (the top-ranked bundles are $(j, \ldots, j)$ and $([j']_{\mathcal{O}_j(K_j)}, [j]_{-\mathcal{O}_j(K_j)})$, for case 1 ($K_{j_1} = 1$) and case 2 ($K_{j_1} > 1$), respectively. If agent j_1 is pessimistic, then for any $d \in D_{i_1}$ with $d \neq j_1$, there exists a bundle whose i_1th component is d and is ranked below any bundle whose i_1th component is j_1. More precisely, if $d \neq Pred_{i_1}(j_1)$, then such an bundle is $([d]_{i_1}, [j]_{-i_1})$; if $d = Pred_{i_1}(j_1)$, then such an bundle is $(d, \ldots, d)$. In both cases a pessimistic agent j_1 will choose item j_1 from D_{i_1}.

Suppose in each round before t, the active agent j always choose item j from the designated category. Let $\mathcal{O}(t) = (j, i)$. If j is optimistic, we show in the following four cases that she will choose item j from D_i in round t.

— $j \neq j_1$, $K_j = 1$. In this case j is guaranteed to get her top-ranked available bundle. It is not hard to check that the available bundles are a subset of BottomBundles_j, where $(j, \ldots, j)$ is available is ranked in the top. Therefore agent j will choose item j.

— $j \neq j_1$, $K_j > 1$. There are following four cases:

(1) $Pred_{i_1}(j)$ has not chosen her item from D_{i_1} and $Pred_{\mathcal{O}_j(K_j)}(j)$ has not chosen her item from $D_{\mathcal{O}_j(K_j)}$. In this case the top-ranked bundle $([Pred_{i_1}(j)]_{i_1}, [j]_{-i_1})$ is still available by the induction hypothesis. We note that $i \neq i_1$ (otherwise item $Pred_{i_1}(j)$ is not available).

(2) $Pred_{i_1}(j)$ has chosen item $Pred_{i_1}(j)$ from D_{i_1} and $Pred_{\mathcal{O}_j(K_j)}(j)$ has not chosen her item from $D_{\mathcal{O}_j(K_j)}$. In this case $([Pred_{i_1}(j)]_{i_1}, [j]_{-i_1})$ is unavailable and $([Pred_{\mathcal{O}_j(K_j)}(j)]_{\mathcal{O}_j(K_j)}, [j]_{-\mathcal{O}_j(K_j)})$ becomes the top available bundle by the induction hypothesis. We note that $i \neq \mathcal{O}_j(K_j)$ (otherwise item $Pred_{\mathcal{O}_j(K_j)}(j)$ is not available).

(3) $Pred_{i_1}(j)$ has chosen item $Pred_{i_1}(j)$ from D_{i_1} and $Pred_{\mathcal{O}_j(K_j)}(j)$ has chosen her item from $D_{\mathcal{O}_j(K_j)}$. In this case, by the induction hypothesis we have that the available bundles are a subset of BottomBundles_j and $(j, \ldots, j)$ is still available and is ranked at the top.

We note it is impossible that $Pred_{i_1}(j)$ has not chosen her item from D_{i_1} and $Pred_{\mathcal{O}_j(K_j)}(j)$ has not chosen her item from $D_{\mathcal{O}_j(K_j)}$, because if so then this means that $(j, \mathcal{O}_j(K_j)) \rhd_{\mathcal{O}} (Pred_{i_1}(j), i_1) \rhd_{\mathcal{O}} (j, i_1)$, which contradicts the definition of K_j. In all three cases above, the ith component of the top-ranked available bundle is j, which means that agent j will choose item j.

— $j = j_1$, $K_j = 1$. By the induction hypothesis, the top-ranked bundle $(j, \ldots, j)$ is still available, which means that agent j will choose item j.

— $j = j_1$, $K_j > 1$. If agent $Pred_{\mathcal{O}_j(K_j)}(j)$ has not chosen her item from $D_{\mathcal{O}_j(K_j)}$, then by the induction hypothesis the top bundle $([Pred_{\mathcal{O}_j(K_j)}(j)]_{\mathcal{O}_j(K_j)}, [j]_{-\mathcal{O}_j(K_j)})$ is still available and $i \neq \mathcal{O}_j(K_j)$. If agent $Pred_{\mathcal{O}_j(K_j)}(j)$ has chosen item $Pred_{\mathcal{O}_j(K_j)}(j)$ from $D_{\mathcal{O}_j(K_j)}$, then by the induction hypothesis the available bundles are a subset of BottomBundles_j with $(j, \ldots, j)$ ranked at the top. In both cases the ith component of the top-ranked available bundle is j. Therefore agent j will choose item j.

If agent j is pessimistic, then by the induction hypothesis the available items in D_i are $D^*_{i,t}$, and $j \in D^*_{i,t}$. For any $d \in D^*_{i,t}$ with $d \neq j$, $([d]_i, [j]_{-i})$ is still available and is ranked lower than any bundle whose ith component is j in R_j. Therefore a pessimistic agent j will choose item j in this round.

It follows that the final allocation is: for all $j \leq n$, agent j gets $(j, \dots, j)$. It is not hard to verify that condition 1 and 2 hold.

To show that condition 3 is also satisfied, consider the allocation where agent j gets $([Pred_{i_1}(j)]_{i_1}, [j]_{-i_1})$. In this allocation, all agents except j_1 get their top-ranked bundle, while j_1 gets her top-ranked bundle (if j_1 is pessimistic) or second-ranked bundle (if j_1 is optimistic). Therefore, condition 3 holds. This proves the theorem.

Table II. Partial preferences for an optimistic agent j. For the cases where $K_j > 1$, we let j' be any agent with $(j, \mathcal{O}_j(K_j - 1)) \rhd_{\mathcal{O}} (j', \mathcal{O}_j(K_j)) \rhd_{\mathcal{O}} (j, \mathcal{O}_j(K_j))$. "Others in BottomBundles$_j$" refers to [BottomBundles$_j \setminus \{(j, \dots, j)\}$].

Optimistic		Order
$j \neq j_1$	case 1: $K_j = 1$	$([Pred_{i_1}(j)]_{i_1}, [j]_{-i_1}) \succ \cdots \succ (j, \dots, j) \succ$ others in BottomBundles$_j$
	case 2: $K_j > 1$	$([Pred_{i_1}(j)]_{i_1}, [j]_{-i_1}) \succ ([Pred_{\mathcal{O}_j(K_j)}(j)]_{\mathcal{O}_j(K_j)}, [j]_{-\mathcal{O}_j(K_j)}) \succ \cdots \succ (j, \dots, j) \succ$ others in BottomBundles$_j$
$j = j_1$	case 1: $K_j = 1$	$(j, \dots, j) \succ ([Pred_{i_1}(j)]_{i_1}, [j]_{-i_1}) \succ$ others
	case 2: $K_j > 1$	$([Pred_{\mathcal{O}_j(K_j)}(j)]_{\mathcal{O}_j(K_j)}, [j]_{-\mathcal{O}_j(K_j)}) \succ ([Pred_{i_1}(j)]_{i_1}, [j]_{-i_1}) \succ \cdots \succ (j, \dots, j) \succ$ others in BottomBundles$_j$

Table III. Partial preferences for a pessimistic agent j. We note that BottomBundles$_j$ is defined differently from that for optimistic agents and for $j = j_1$, "others in BottomBundles$_j$" refers to [BottomBundles$_j \setminus \{(j, \dots, j), ([Pred_{i_1}(j)]_{i_1}, [j]_{-i_1})\}$].

Pessimistic	Order
$j \neq j_1$	$([Pred_{i_1}(j)]_{i_1}, [j]_{-i_1}) \succ \cdots \succ (j, \dots, j) \succ$ others in BottomBundles$_j$
$j = j_1$	$([Pred_{i_1}(j)]_{i_1}, [j]_{-i_1}) \succ \cdots \succ (j, \dots, j) \succ$ others in BottomBundles$_j \succ (Pred_{i_1}(j), \dots, Pred_{i_1}(j))$