X

Allocating Indivisible Items in Categorized Domains

Lirong Xia, Computer Science Department
Rensselaer Polytechnic Institute
110 8th Street, Troy, NY 12180, USA
xial@cs.rpi.edu

To overcome the communicational and computational barriers in allocation problems of indivisible items, we propose a novel and general class of allocation problems called *categorized domain allocation problems (CDAPs)*, where the indivisible items are partitioned into multiple categories and we must allocate the items to the agents without monetary transfer, such that each agent gets at least one item from each category.

We focus on the design and analysis of allocation mechanisms for *basic CDAPs*, where the number of items in each category is exactly the same as the number of agents. We start with serial dictatorships and characterize them by a minimal set of three properties: *strategy-proofness*, *non-bossiness*, and *category-wise neutrality*. Then, we design and analyze a natural extension of serial dictatorships called *categorical sequential allocation mechanisms*, which allocate the items in multiple rounds so that in each round, the active agent chooses an item from the designated category. We characterize the worst-case ordinal efficiency of categorical sequential allocation mechanisms for optimistic and pessimistic agents, and use computer simulations to study the expected efficiency of these mechanisms.

Categories and Subject Descriptors: J.4 [**Computer Applications**]: Social and Behavioral Sciences—*Economics*; I.2.11 [**Distributed Artificial Intelligence**]: Multiagent Systems

General Terms: Algorithms, Economics, Theory

1. INTRODUCTION

Suppose we must allocate 10 discussion topics and 10 dates to 10 students to organize a seminar. Students may have different and combinatorial preferences over (topic, date) bundles, and their preferences over one component may depend on the other component. For example, it is quite possible that if a student gets an easy topic, then she would prefer to take an early date to better enjoy the rest of the seminar; but if she gets a hard topic, then she would prefer to take a late date for better preparation.

This example illustrates a novel setting of allocating multiple indivisible items, which we call *categorized domains*. A categorized domain contains multiple indivisible items and each item belongs to one of the $p \geq 1$ categories. In *categorized domain allocation problems (CDAPs)*, we want to design a mechanism to allocate items in a categorized domain to agents, such that each agent gets at least one item from each category, and no monetary transfer is allowed. In the above example, there are two categories of items: topics and dates, and each agent (student) must get a topic and a date to lead the discussion.

Let us give a more complicated example of CDAP. Consider a generic multi-agent setting of allocating tasks and equipments to agents, e.g. allocating anti-terror tasks and equipments to a group of SWAT troopers. There are two categories of "items": the tasks and the equipments, and it is possible that a trooper gets more than one task or more than one equipment. Moreover, both tasks and equipments may be further divided into multiple sub-categories, so that the number of categories may be well beyond a few. For example, it is often the case that each trooper must get at least one of firearms, body armors, ballistic shields, entry tools, etc.

In general, non-categorized domains, the design and analysis of mechanisms to allocate indivisible items without money have constituted an active research area at the interface of computer science and economics. On the computer science side, allocation problems have been studied under the active research area known as *centralized multiagent resource allocation* [Chevaleyre et al. 2006], where the number of items can be much larger than the number of agents. The main research agenda is to tackle computational challenges in preference representation and communication, and the computation of allocations that maximize various kinds of *social welfare* and *fairness*. On the economics side, the allocation problems have been studied under the research stream called *one-sided matching*, also known as *assignment problems* [Sönmez and Ünver 2011]. Most previous research focused on simple, nicely structured, and practical domains, for example assigning n houses to n families. The main objective is to design allocation mechanisms with desired economic properties, especially strategy-proofness and good welfare properties.

Nevertheless, previous research has been challenged and hindered by the following barriers.

• *Preference bottleneck:* When the number of items is not too small, it is impractical for the agents to explicitly report their preferences over all 2^m bundles of m items, either ordinally by a full ranking over all 2^m bundles, or cardinally by giving one number (utility) for each of the 2^m bundles.

• *Computational bottleneck:* Even if the agents can report their preferences compactly, computing an "optimal" allocation is often a hard *combinatorial optimization problem* [Papadimitriou and Steiglitz 1998].

• *Threats of agents' strategic behavior:* An agent may have incentive to report untruthful preferences to make her own allocation better off. This may lead to a socially undesirable allocation.

1.1. Our Contributions

We propose to work towards breaking the three aforementioned barriers in the novel setting of categorized domain allocation problems. CDAPs are a general setting since many real-world allocation problems are CDAPs as we showed in the examples above, and non-categorized allocation problems can be seen as CDAPs with just one category. CDAPs are our main conceptual contribution.

As an illustration of the viability of the CDAP framework, in this paper we focus on the design and analysis of novel mechanisms for a special yet still quite general class of CDAPs, called *basic categorized domain allocation problems (basic CDAPs)*. In a basic CDAP, the number of items in each category is exactly the same as the number of agents, as in the seminar-organization example shown in the beginning. Agents' preferences are represented by linear orders over bundles of items. Hence, we want to find an allocation so that each agent gets exactly one item from each category.

To analyze and overcome the threats of agents' strategic behavior, we investigate the possibility of strategy-proof mechanisms from a classical mechanism design point of view, by ignoring the communicational complexity for the moment, and allowing agents to report their preferences in full to the center. For basic CDAPs with at least two categories, we give a characterization of *serial dictatorships* by a minimal set of three normative properties: *strategy-proofness*, *non-bossiness*, and *category-wise neutrality*.

To overcome the preference bottleneck and the computational bottleneck and go beyond serial dictatorships, we move on to design indirect mechanisms, where agents do not directly report their preferences (linear orders over all bundles) to the center in full. We propose and analyze a class of indirect, distributed mechanisms called *categorical sequential allocation mechanisms*, which are natural extensions of serial dictatorships, sequential allocation protocols [Bouveret and Lang 2011], and the draft mechanism [Budish and Cantillon 2012] for non-categorized domains. For n agents and p categories, a categorical sequential mechanism is defined by an ordering over all (agent,category) pairs, such that in each round $t = 1, \ldots, np$, the active agent picks an item from the designated category. We analyze the worst-case ordinal efficiency of categorical sequential mechanisms for two types of altruistic and myopic agents: *optimistic* agents, who always choose the item in their top-ranked bundle that is still available, and *pessimistic* agents, who always choose the item that gives her optimal worst-case guarantee.[1] We then apply this characterization to compare various categorical sequential allocations, and observe that while serial dictatorships with all-optimistic agents have the best worst-case utilitarian rank, they have the worst worst-case egalitarian rank. On the other hand, the balanced mechanisms with all-pessimistic agents have good worst-case utilitarian rank.

We use computer simulations to compare various categorical sequential allocation mechanisms w.r.t. the expected utilitarian rank and expected egalitarian rank. We use the Mallows model [Mallows 1957] with different dispersion parameters to generate agents' preferences. Our results indicate that among the mechanisms we choose to compare, serial dictatorships with optimistic agents have the best expected utilitarian rank, and balanced mechanisms with pessimistic agents have the best expected egalitarian rank.

1.2. Discussions and Related Work

We are not aware of previous work studying allocation problems in categorized domains. On the modeling level, CDAPs can be seen as standard centralized multi-agent resource allocation problems without the categorical information. However, directly applying generic techniques in centralized multi-agent resource allocation and one-sided matching does not help with overcoming the three barriers, especially the preference bottleneck and computational bottleneck. As we will see later in this paper, considering the categorical information creates more possibilities of design, including the categorical sequential mechanisms we study in this paper. More importantly, we believe that CDAPs are a natural and general framework to apply techniques and ideas developed in other fields on preference representation and aggregation. For example, the categorical information facilitates the design of richer and more natural ordinal preference language that captures agents' conditional preferences between categories as in CP-nets [Boutilier et al. 2004]. It also bridges CDAPs and combinatorial voting [Brandt et al. 2013].

On the technical level, typical one-sided matching problems are basic CDAPs with one category. Our characterization of serial dictatorships for basic CDAPs may look similar to some characterizations of serial dictatorships and similar mechanisms for different models studied in one-sided matching. However, we do not see a way to extend previous characterization to the novel setting of categorized domains, and the categorical information plays a critical role in our characterization and proofs. Below we briefly discuss other characterizations and show how ours is different.

Svensson [1999] characterized serial dictatorships for non-categorized domains, where each agent must gets exactly one item. This setting can be seen as the basic CDAP with one category, and our characterization is for basic CDAP with $p \geq 2$ categories. The setting of [Pápai 2000a] is also restricted to each agent getting exactly one item. Pápai [2000b, 2001] studied the setting where each agent can get multiple items, but there is no categorical constraints on the bundles allocated to the agents, and agents are allowed to get nothing. The setting of [Ehlers and Klaus 2003] assumes that agents' preferences over bundles are induced by their preferences over single items. The characterization by Hatfield [2009] assumes that agents's preferences

[1]Similar types of agents have been studied to analyze mechanisms in other social choice setting. For example, *paradoxes in multiple elections* arise when agents are optimistic and vote for their top-ranked alternative [Brams et al. 1998; Lacy and Niou 2000]. Pessimistic agents naturally correspond to the *maximin* agents studied by Brams et al. [2006].

over bundles are represented by additive utilities over single items. Our focus of this paper (basic CDAPs with multiple categories) is not a generalization of all aforementioned settings, since none of them can model the categorical constraint: each agent must get exactly p items, one from each category. This makes directly extending previous characterizations to basic CDAPs hard, if not impossible.

Our analysis of the worst-case ordinal efficiency of categorical sequential allocation mechanisms resembles the *price of anarchy* [Koutsoupias and Papadimitriou 1999], which is defined for strategic and self-interested agents, with the presence of a social welfare function to numerically evaluate the outcomes. Our theorem is also related to the notion of *distortion* in the voting setting [Procaccia and Rosenschein 2006; Boutilier et al. 2012], which concerns the social welfare loss by reporting a ranking instead of a utility function. However, our theorem is significantly different because we focus on allocation problems, and we do not need to assume the existence of agents' cardinal utilities and a social welfare function, even though our theorem can be easily extended to study worst-case social welfare loss given a social welfare function, as we will see in Proposition 2 through Proposition 5.

2. CATEGORIZED DOMAINS AND THE ALLOCATION PROBLEMS

We start with the definitions of general categorized domains, the corresponding allocation problems, and the special cases which we will focus on in the rest of this paper.

Definition 1 *A* categorized domain *is a set of indivisible items partitioned into $p \geq 1$ categories $\{D_1, \dots, D_p\}$. In a* categorized domain allocation problem (CDAP)*, we want to allocate the items to the agents, such that each agent gets at least one item from each category, and no monetary transfer is allowed.*

In a basic categorized domain *for n agents, for each $i \leq p$, $D_i = \{1_i, \dots, n_i\}$, $\mathfrak{D} = D_1 \times \cdots \times D_p$, and each agent's preferences are represented by a linear order over $\mathfrak{D}$. In a* basic categorized domain allocation problem (basic CDAP)*, we want to allocate all items to the agents, such that every agent gets exactly one bundle in $\mathfrak{D}$.*

In the rest of this paper we focus on basic categorized domains and basic CDAPs. For simplicity, the subscripts in $\{1_i, \dots, n_i\}$ are often omitted and we write $D_i = \{1, \dots, n\}$. For any $j \leq n$, let R_j denote a linear order over $\mathfrak{D}$ and let $P = (R_1, \dots, R_n)$ denote a *preference profile*. An *allocation* A is a mapping from $\{1, \dots, n\}$ to $\mathfrak{D}$, such that for every $j \leq n$, $A(j)$ is the bundle assigned to agent j, which means that for any $i \leq p$, we have $\cup_{j=1}^{n}[A(j)]_i = D_i$, where $[A(j)]_i$ is the i-th component of the bundle allocated to agent j, i.e., the item in category i in the allocation to agent j. An *allocation mechanism* (or *mechanism* for short) f is a mapping that takes a preference profile as input, and outputs an allocation. In this paper we sometimes use $f^j(P)$ to denote $(f(P))(j)$, that is, the bundle allocated to agent j when the preference profile is P.

We say a direct mechanism f satisfies *strategy-proofness*, if no self-interested agent can benefit from misreporting her preferences. f satisfies *non-bossiness*, if no agent is *bossy* in f. An agent is bossy if she can report differently to change the bundles allocated to some other agents, while keeping her own allocation unchanged. f satisfies *category-wise neutrality*, if after applying any permutation that only permutes the names of items within the same category, the allocation is also permuted in the same way. Formally, we have the following definition.

Definition 2 *A mechanism f satisfies* strategy-proofness*, if for any preference profile P, any agent j, and any linear order R'_j over $\mathfrak{D}$, $f^j(P) \succ_{R_j} f^j(R'_j, R_{-j})$. f satisfies* non-bossiness*, if for any preference profile P, any agent j, and any linear order R'_j over $\mathfrak{D}$, $[f^j(P) = f^j(R'_j, R_{-j})] \Rightarrow [f(P) = f(R'_j, R_{-j})]$. f satisfies* category-wise neutrality*, if* *for any preference profile P, any category i, and any permutation M_i over D_i, we have $f(M_i(P)) = M_i(f(P))$, where for any bundle $\vec{d} \in \mathfrak{D}$, $M_i(\vec{d}) = (M_i([\vec{d}]_i), [\vec{d}]_{-i})$.*

When there is only one category, category-wise neutrality degenerates to the traditional neutrality used in characterizations of strategy-proof allocation mechanisms for non-categorized domains, e.g. by Svensson [1999]. When the number of categories is more than one, category-wise neutrality is much weaker than the traditional neutrality.

A mechanism is a *serial dictatorship*, if there exists a linear order $\mathcal{K}$ over $\{1, \ldots, n\}$ such that for any preference profile P, agents choose their top-ranked available bundle sequentially according to $\mathcal{K}$. We note that any serial dictatorship can be viewed as a *distributed protocol*, where agents do not explicitly report their preferences to the center. In this section and the next section, we still view serial dictatorships as direct mechanisms.

Example 1 *Suppose there are* 3 *agents and* 2 *categories.* $\mathfrak{D} = \{1, 2, 3\} \times \{1, 2, 3\}$. *Agents' preferences over the* 9 *bundles are the following.*

Agent 1: $R_1 = 12 \succ 21 \succ 32 \succ 33 \succ 31 \succ 22 \succ 23 \succ 13 \succ 11$
Agent 2: $R_2 = 32 \succ 12 \succ 21 \succ 13 \succ 33 \succ 11 \succ 31 \succ 23 \succ 22$
Agent 3: $R_3 = 13 \succ 12 \succ 11 \succ 22 \succ 32 \succ 21 \succ 33 \succ 31 \succ 23$

Let us apply the serial dictatorship with $\mathcal{K} = 1 \rhd 2 \rhd 3$.[2] *Suppose agents report their preferences truthfully. In the first round,* 12 *is allocated to agent* 1. *In the second round, agent* 2 *cannot get* 32 *or* 12 *since item* 2 *is unavailable. So* 21 *is allocated to agent* 2. *In the final round, agent* 3 *is left with* 33.

3. AN AXIOMATIC CHARACTERIZATION OF SERIAL DICTATORSHIPS FOR BASIC CATEGORIZED DOMAINS

In this section we characterize serial dictatorship by strategy-proofness, non-bossiness, and category-wise neutrality, for basic categorized domains with at least two categories. We also show that the three properties are minimal for characterizing serial dictatorships in basic categorized domains.

Theorem 1 *For any* $p \geq 2$ *and* $n \geq 2$, *an allocation mechanism for basic categorized domain is strategy-proof, non-bossy, and category-wise neutral if and only if it is a serial dictatorship. Moreover, strategy-proofness, non-bossiness and category-wise neutrality are a minimal set of properties that characterize serial dictatorships.*

Proof: The proof is inspired by proofs in [Pápai 2000b, 2001; Hatfield 2009] but we do not see an easy way to extend their proofs to categorized domains. We first prove four lemmas. The first three lemmas are standard in proving characterizations for serial dictatorships and their proofs can be found in the Appendix. The last one (Lemma 4) is new, whose proof is the most involved and heavily relies on the categorical information.

The first lemma (roughly) says that for all strategy-proof and non-bossy mechanism f and all preference profile P, if every agent j reports a different ranking without enlarging the set of bundles ranked above $f^j(P)$ (and she can shuffle the bundles ranked above $f^j(P)$ and she can shuffle the bundles ranked below $f^j(P)$), then the allocation to all agents does not change in the new preference profile. This resembles *(strong) monotonicity* in social choice.

[2]In this paper we use $\rhd$ to denote a linear order over agents or a linear order over (agent,category) pairs to distinguish from agents' preferences $\succ$ over bundles.

Lemma 1 *Let f be a strategy-proof and non-bossy allocation mechanism over a basic categorized domain with $p \geq 2$. For any pair of preference profiles P and P' such that for all $j \leq n$, $\{\vec{d} \in \mathfrak{D} : \vec{d} \succ_{R'_j} f^j(P)\} \subseteq \{\vec{d} \in \mathfrak{D} : \vec{d} \succ_{R_j} f^j(P)\}$, we have $f(P') = f(P)$.*

For any linear order R over $\mathfrak{D}$ and any bundle $\vec{d} \in \mathfrak{D}$, we say a linear order R' is a *pushup* of $\vec{d}$ from R, if R' can be obtained from R by raising the position of $\vec{d}$ while keeping the relative positions of other bundles unchanged. The next lemma states that for any strategy-proof and non-bossy mechanism f, if an agent reports her preferences differently by only pushing up a bundle $\vec{d}$, then either the allocation to all agents does not change, or she gets $\vec{d}$.

Lemma 2 *Let f be a strategy-proof and non-bossy allocation mechanism over a basic categorized domain with $p \geq 2$. For any profile P, any $j \leq n$, any bundle $\vec{d}$, and any R'_j that is a pushup of $\vec{d}$ from R_j, either (1) $f(R'_j, R_{-j}) = f(R)$ or (2) $f^j(R'_j, R_{-j}) = \vec{d}$.*

We next prove that strategy-proofness, non-bossiness, and category-wise neutrality altogether imply *Pareto-optimality*, which states that for any preference profile P, there does not exists an allocation A such that all agents prefer their bundles in A than their bundles in $f(P)$, and some of them strictly prefer their bundles in A.

Lemma 3 *For any basic categorized domains with $p \geq 2$, any strategy-proof, non-bossy, and category-wise neutral allocation mechanism is Pareto optimal.*

The next lemma states that for any strategy-proof and non-bossy allocation mechanism f, any preference profile P, and any pair of agents j_1, j_2, there is no bundle $\vec{c}$ that only contains items allocated to agent j_1 and j_2 by f, such that both j_1 and j_2 prefer $\vec{c}$ to their bundles allocated by f.

Lemma 4 *Let f be a strategy-proof and non-bossy allocation mechanism over a basic categorized domain with $p \geq 2$. For any preference profile P and any $j_1 \neq j_2 \leq n$, let $\vec{a} = f^{j_1}(P)$ and $\vec{b} = f^{j_2}(P)$, there does not exist $\vec{c} \in \{a_1, b_1\} \times \{a_2, b_2\} \times \cdots \times \{a_p, b_p\}$ such that $\vec{c} \succ_{R_{j_1}} \vec{a}$ and $\vec{c} \succ_{R_{j_2}} \vec{b}$, where a_i is the i-th component of $\vec{a}$.*

Proof: Suppose for the sake of contradiction that such a bundle $\vec{c}$ exists. Let $\vec{d}$ denote the bundle such that $\vec{c} \cup \vec{d} = \vec{a} \cup \vec{b}$. More precisely, for all $i \leq m$, $\{c_i, d_i\} = \{a_i, b_i\}$. For example, if $\vec{a} = 1213$, $\vec{b} = 2431$, and $\vec{c} = 1211$, then $\vec{d} = 2433$.

The rest of the proof derives a contradiction by proving the a series of observations as illustrated in Table I. In each step, we prove that the boxed bundles are allocated to agent j_1 and agent j_2 respectively, and all other agents get their top-ranked bundles.

Table I. Proof sketch of Lemma 4.

$\hat{R}_{j_1} : \vec{c} \succ \boxed{\vec{a}} \succ \vec{d} \succ \vec{b} \succ$ others $\hat{R}_{j_2} : \vec{c} \succ \boxed{\vec{b}} \succ \vec{a} \succ \vec{d} \succ$ others Other j : $f^j(P) \succ$ others	$\hat{R}_{j_1} : \vec{c} \succ \boxed{\vec{a}} \succ \vec{d} \succ \vec{b} \succ$ others $\bar{R}_{j_2} : \vec{c} \succ \vec{a} \succ \boxed{\vec{b}} \succ \vec{d} \succ$ others Other j : $f^j(P) \succ$ others	$\bar{R}_{j_1} : \vec{c} \succ \boxed{\vec{b}} \succ \vec{a} \succ \vec{d} \succ$ others $\bar{R}_{j_2} : \vec{c} \succ \boxed{\vec{a}} \succ \vec{b} \succ \vec{d} \succ$ others Other j : $f^j(P) \succ$ others
Step 1	Step 2	Step 3
$\bar{R}_{j_1} : \vec{c} \succ \boxed{\vec{b}} \succ \vec{a} \succ \vec{d} \succ$ others $\mathring{R}_{j_2} : \vec{c} \succ \boxed{\vec{a}} \succ \vec{d} \succ \vec{b} \succ$ others Other j : $f^j(P) \succ$ others	$\mathring{R}_{j_1} : \vec{c} \succ \vec{a} \succ \boxed{\vec{b}} \succ \vec{d} \succ$ others $\mathring{R}_{j_2} : \vec{c} \succ \boxed{\vec{a}} \succ \vec{d} \succ \vec{b} \succ$ others Other j : $f^j(P) \succ$ others	$\mathring{R}_{j_1} : \vec{c} \succ \boxed{\vec{a}} \succ \vec{b} \succ \vec{d} \succ$ others $\bar{R}_{j_2} : \vec{c} \succ \vec{a} \succ \boxed{\vec{b}} \succ \vec{d} \succ$ others Other j : $f^j(P) \succ$ others
Step 4	Step 5	Step 6

Step 1. Let $\hat{R}_{j_1} = [\vec{c} \succ \vec{a} \succ \vec{d} \succ \vec{b} \succ \text{others}]$, $\hat{R}_{j_2} = [\vec{c} \succ \vec{b} \succ \vec{a} \succ \vec{d} \succ \text{others}]$, where "others" represents an arbitrary linear order over the remaining bundles, and for any $j \neq j_1, j_2$, let $\hat{R}_j = [f^j(P) \succ \text{others}]$. By Lemma 1, $f(\hat{P}) = f(P)$.

Step 2. Let $\bar{R}_{j_2} = [\vec{c} \succ \vec{a} \succ \vec{b} \succ \vec{d} \succ \text{others}]$ be a pushup of $\vec{a}$ from $\hat{R}_{j_2}$. We will prove that $f(\bar{R}_{j_2}, \hat{R}_{-j_2}) = f(\hat{P}) = f(P)$. Since $\bar{R}_{j_2}$ is a pushup of $\vec{a}$ from $\hat{R}_{j_2}$, by Lemma 2, $f^{j_2}(\bar{R}_{j_2}, \hat{R}_{-j_2})$ is either $\vec{a}$ or $\vec{b}$. We now show that the former case is impossible. Suppose for the sake of contradiction $f^{j_2}(\bar{R}_{j_2}, \hat{R}_{-j_2}) = \vec{a}$, then $f^{j_1}(\bar{R}_{j_2}, \hat{R}_{-j_2})$ cannot be $\vec{c}$, $\vec{a}$, or $\vec{d}$ since otherwise some item will be allocated twice. This means that $f(\bar{R}_{j_2}, \hat{R}_{-j_2})$ is Pareto dominated by the allocation where j_1 gets $\vec{d}$, j_2 gets $\vec{c}$, and all other agents get their top-ranked bundles. This contradicts the Pareto-optimality of f (Lemma 3). Hence $f^{j_2}(\bar{R}_{j_2}, \hat{R}_{-j_2}) = \vec{b} = f^{j_2}(\hat{P})$. By non-bossiness we have $f(\bar{R}_{j_2}, \hat{R}_{-j_2}) = f(\hat{P}) = f(P)$.

Step 3. Let $\bar{R}_{j_1} = [\vec{c} \succ \vec{b} \succ \vec{a} \succ \vec{d} \succ \text{others}]$ be a pushup of $\vec{b}$ from $\hat{R}_{j_1}$. We will prove that in $f(\bar{R}_{j_1}, \bar{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}})$, j_1 gets $\vec{b}$, j_2 gets $\vec{a}$, and all other agents get the same items as in $f(P)$. Since $\bar{R}_{j_1}$ is a pushup of $\vec{b}$ from $\hat{R}_{j_1}$, by Lemma 2, $f^{j_1}(\bar{R}_{j_1}, \bar{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}})$ is either $\vec{a}$ or $\vec{b}$. We now show that the former case is impossible. Suppose for the sake of contradiction that $f^{j_1}(\bar{R}_{j_1}, \bar{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}}) = \vec{a}$. By non-bossiness, $f^{j_2}(\bar{R}_{j_1}, \bar{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}}) = \vec{b}$. This means that $f(\bar{R}_{j_1}, \bar{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}})$ is Pareto-dominated by the allocation where j_1 gets $\vec{b}$, j_2 gets $\vec{a}$, and all other agents get their top-ranked bundles. This contradicts the Pareto-optimality of f (Lemma 3).

Step 4. Let $\mathring{R}_{j_2} = [\vec{c} \succ \vec{a} \succ \vec{d} \succ \vec{b} \succ \text{others}]$ be a pushup of $\vec{d}$ from $\bar{R}_{j_2}$. By Lemma 1, $f(\bar{R}_{j_1}, \mathring{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}}) = f(\bar{R}_{j_1}, \bar{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}})$.

Step 5. Let $\mathring{R}_{j_1} = [\vec{c} \succ \vec{a} \succ \vec{b} \succ \vec{d} \succ \text{others}]$ be a pushup of $\vec{a}$ from $\bar{R}_{j_1}$. We will prove that $f(\mathring{R}_{j_1}, \mathring{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}}) = f(\bar{R}_{j_1}, \mathring{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}})$. Since $\mathring{R}_{j_1}$ is a pushup of $\vec{a}$ from $\bar{R}_{j_1}$, by Lemma 2, $f^{j_1}(\mathring{R}_{j_1}, \mathring{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}})$ is either $\vec{a}$ or $\vec{b}$. We now show that the former case is impossible. Suppose for the sake of contradiction that $f^{j_1}(\mathring{R}_{j_1}, \mathring{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}}) = \vec{a}$. Then in $f(\mathring{R}_{j_1}, \mathring{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}})$, agent j_2 cannot get $\vec{c}$, $\vec{a}$, or $\vec{d}$, which means that $f(\mathring{R}_{j_1}, \mathring{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}})$ is Pareto-dominated by the allocation where j_1 gets $\vec{c}$, j_2 gets $\vec{d}$, and all other agents get their top-ranked bundles. This contradicts the Pareto-optimality of f. Hence, $f^{j_1}(\mathring{R}_{j_1}, \mathring{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}}) = \vec{b}$. By non-bossiness $f(\mathring{R}_{j_1}, \mathring{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}}) = f(\bar{R}_{j_1}, \mathring{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}})$.

Step 6. We note that $\mathring{R}_{j_1}$ is a pushup of $\vec{b}$ from $\hat{R}_{j_1}$ (and $\vec{b}$ is still below $\vec{a}$). By Lemma 1, $f(\mathring{R}_{j_1}, \bar{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}}) = f(\hat{R}_{j_1}, \bar{R}_{j_2}, \hat{R}_{-\{j_1,j_2\}})$. We note that the right hand side is the profile in Step 2.

Contradiction. Finally, the observations in Step 5 and Step 6 imply that when agents' preferences are as in Step 6, agent j_2 has incentive to report $\mathring{R}_{j_2}$ in Step 5 to improve the bundle allocated to her (from $\vec{b}$ to $\vec{a}$). This contradicts the strategy-proofness of f and completes the proof of Lemma 4. ∎

(Continuing the proof of Theorem 1). It is easy to check that any serial dictatorship satisfies strategy-proofness, non-bossiness and category-wise neutrality. We now prove that any mechanism satisfying the three properties must be a serial dictatorship. Let R be a linear order over $\mathfrak{D}$ that satisfies the following conditions.

- $(1, \ldots, 1) \succ (2, \ldots, 2) \succ \cdots \succ (n, \ldots, n)$.

• For any $j < n$, the bundles ranked between $(j, \ldots, j)$ and $(j+1, \ldots, j+1)$ are those satisfying the following two conditions: 1) at least one component is j, and 2) all components are in $\{j, j+1, \ldots, n\}$. Let B_j denote these bundles. That is, $B_j \subseteq \mathfrak{D}$ and $B_j = \{\vec{d} : \forall l, d_l \geq j \text{ and } \exists l', d_{l'} = j\}$.

• For any i and any $\vec{d}, \vec{e} \in B_j$, if the number of j's in $\vec{d}$ is strictly larger than the number of j's in $\vec{e}$, then $\vec{d} \succ \vec{e}$.

Let $P = (R, \ldots, R)$. We prove the following claim.

Claim 1 *For any $l \leq n$, there exists $j_l \leq n$ such that $f^{j_l}(P) = (l, \ldots, l)$.*

Proof: The claim is proved by induction on l. When $l = 1$. For the sake of contradiction suppose there is no j_l with $f^{j_l}(P) = (1, \ldots, 1)$. Then there exist a pair of agents j and j' such that both $\vec{a} = f^j(P)$ and $\vec{b} = f^{j'}(P)$ contain 1 in at least one category.

Let $\vec{c}$ be the bundle obtained from $\vec{a}$ by replacing items in categories where $\vec{b}$ takes 1 to 1. More precisely, we let $\vec{c} = (c_1, \ldots, c_p)$ s.t. $c_i = \begin{cases} 1 & \text{if } a_i = 1 \text{ or } b_i = 1 \\ a_i & \text{otherwise} \end{cases}$.

It follows that in R, $\vec{c} \succ_R \vec{a}$ and $\vec{c} \succ_R \vec{b}$ since the number of 1's in $\vec{c}$ is strictly larger than the number of 1's in $\vec{a}$ or $\vec{b}$. By Lemma 4, this contradicts the assumption that f is strategy-proof and non-bossy. Hence there exists $j_1 \leq n$ with $f^{j_1}(P) = (1, \ldots, 1)$.

Suppose the claim is true for $l \leq l'$. We next prove that there exists $j_{l'+1}$ such that $f^{j_{l'+1}}(P) = (l'+1, \ldots, l'+1)$. This follows after a similar reasoning to the $l = 1$ case. Formally, suppose for the sake of contradiction there does not exist such a $j_{l'+1}$. Then, there exist two agents who get $\vec{a}$ and $\vec{b}$ in $f(P)$ such that both $\vec{a}$ and $\vec{b}$ contain $l'+1$ in at least one category. By the induction hypothesis, items $\{1, \ldots, l'\}$ in all categories have been allocated, which means that all components of $\vec{a}$ and $\vec{b}$ are at least as large as $l'+1$. Let $\vec{c}$ be the bundle obtained from $\vec{a}$ by replacing items in all categories where $\vec{b}$ takes $l'+1$ to $l'+1$. We have $\vec{c} \succ_R \vec{a}$ and $\vec{c} \succ_R \vec{b}$, leading to a contradiction by Lemma 4. Therefore, the claim holds for $l = l'+1$. This completes the proof of Claim 1. ■

Back to the proof of the theorem, w.l.o.g. we let $j_1 = 1$, $j_2 = 2$, …, $j_n = n$ denote the agents in Claim 1. For any profile $P' = (R'_1, \ldots, R'_n)$, we define n bundles as follows. Let $\vec{d^1}$ denote the top-ranked bundle in R'_1, and for any $l \geq 2$, let $\vec{d^l}$ denote agent l's top-ranked available bundle given that items in $\vec{d^1}, \ldots, \vec{d^{l-1}}$ have already been allocated. That is, $\vec{d^l}$ is the most preferred bundle in $\{\vec{d} : \forall l' < l, \vec{d} \cap \vec{d^{l'}} = \emptyset\}$ according to R'_l. Then, for any $i \leq m$, we define a category-wise permutation M_i such that for all $l \leq n$, $M_i(l) = [\vec{d^l}]_i$, where we recall that $[\vec{d^l}]_i$ is the item in the i-th category in $\vec{d^l}$. Let $M = (M_1, \ldots, M_m)$. It follows that for all $l \leq n$, $M(l, \ldots, l) = \vec{d^l}$. By category-wise neutrality and Claim 1, in $f(M(P))$ agent l gets $M(f^l(P)) = \vec{d^l}$.

Comparing $M(P)$ with P', we notice that for all $l \leq n$ and all bundle $\vec{e}$, if $\vec{d^l} \succ_{M(R)} \vec{e}$ then $\vec{d^l} \succ_{R'_l} \vec{e}$. This is because if there exists $\vec{e}$ such that $\vec{d^l} \succ_{M(R)} \vec{e}$ but $\vec{e} \succ_{R'_l} \vec{d^l}$, then $\vec{e}$ is still available after $\{\vec{d^1}, \ldots, \vec{d^{l-1}}\}$ have been allocated, and $\vec{e}$ is ranked higher than $\vec{d^l}$ in R'_l. This contradicts the selection of $\vec{d^l}$. By Lemma 1, $f(P') = f(M(P)) = M(f(P))$, which proves that f is the serial dictatorship w.r.t. the order $1 \rhd 2 \rhd \cdots \rhd n$.

Next, we show that strategy-proofness, non-bossiness, and category-wise neutrality are a minimal set of properties that characterize serial dictatorships.

strategy-proofness is necessary: Consider the allocation mechanism that maximizes the social welfare w.r.t. the following utility functions. For any $i \leq n^p$ and $j \leq n$, the bundle ranked at the i-th position in agent j's preferences gets $(n^p - i)(1 + (\frac{1}{2n^p})^j)$

points.[3] It is not hard to check that for any pair of different allocations, the social welfares are different. It follows that this allocation mechanism satisfies non-bossiness. This is because if agent j's allocation is the same when only she reports differently, then the set of items left to the other agents is the same, which means that the allocation to the other agents by the mechanism is the same. Since the utility of a bundle only depends on its position in the agents' preferences rather than the name of the bundle, the allocation mechanism satisfies category-wise neutrality. This mechanism is not a serial dictatorship. To see this, consider the basic categorized domain with $p = n = 2$, $R'_1 = [11 \succ 12 \succ 22 \succ 21]$, and $R'_2 = [12 \succ 21 \succ 11 \succ 22]$. A serial dictatorship will either give 11 to agent 1 and give 22 to agent 2, or give 21 to agent 1 and give 12 to agent 2, but the allocation that maximizes social welfare w.r.t. the utility function described above is to give 12 to agent 1 and give 21 to agent 2.

non-bossiness is necessary: Consider the following "conditional serial dictatorship": agent 1 always chooses her favorite bundle in the first round, and the order over the remaining agents $\{2, \dots, n\}$ depends on agent 1's preferences in the following way: if the first component of agent 1's second-ranked bundle is the same as the first component of her top choice, then the order over the rest of agents is $2 \rhd 3 \rhd \cdots \rhd n$; otherwise it is $n \rhd n-1 \rhd \cdots \rhd 2$. It is not hard to verify that this mechanism satisfies strategy-proofness and category-wise neutrality, and is not a serial dictatorship (where the order must be fixed before seeing the preference profile).

category-wise neutrality is necessary: Consider the following "conditional serial dictatorship": agent 1 always chooses her favorite bundle in the first round, and the order over agents $\{2, \dots, n\}$ depends on the allocation of agent 1 in the following way: if agent 1 gets $(1, \dots, 1)$, then the order over the rest of agents is $2 \rhd 3 \rhd \cdots \rhd n$; otherwise it is $n \rhd n-1 \rhd \cdots \rhd 2$. It is not hard to verify that this mechanism satisfies strategy-proofness and non-bossiness, and is not a serial dictatorship. ■

4. CATEGORICAL SEQUENTIAL ALLOCATION MECHANISMS

In this section we extend serial dictatorships to define *categorical sequential allocation mechanisms* (*categorical sequential mechanisms* for short) for basic categorized domains.

Given a linear order $\mathcal{O}$ over $\{1, \dots, n\} \times \{1, \dots, p\}$, the categorical sequential mechanism $f_{\mathcal{O}}$ allocates the items in np steps as illustrated in Algorithm 1. In each step t, suppose the t-th element in $\mathcal{O}$ is (j, i), (equivalently, $t = \mathcal{O}^{-1}(j, i)$). Agent j is called the *active agent* in step t and she must choose an available item from D_i (meaning that no agent has chosen that item before round t), denoted by $d_{j,i}$. Then, $d_{j,i}$ is broadcast to all agents and we move on to the next step.

We emphasize that in categorical sequential mechanisms, in each step the active agent must choose an item from the designated category. Hence, categorical sequential mechanism are different from sequential allocation protocols [Bouveret and Lang 2011] and the draft mechanism [Budish and Cantillon 2012], where in each step the active agent can choose any single available item. We now give an example of two categorical sequential mechanisms.

Example 2 *Any serial dictatorship with order $\mathcal{K} = j_1 \rhd \cdots \rhd j_n$ is a categorial sequential mechanism with the following order:* $(j_1, 1) \rhd (j_1, 2) \rhd \cdots \rhd (j_1, p) \rhd \cdots \rhd (j_n, 1) \rhd (j_n, 2) \rhd \cdots \rhd (j_n, p)$.

[3]The $(\frac{1}{2n^p})^j$ terms in the utility functions are only used to avoid ties in allocations. In fact, any utility functions where there are no ties satisfy non-bossiness and category-wise neutrality, but some of them are equivalent to serial dictatorships, which are the cases we want to avoid in our proof.

Algorithm 1: Categorical sequential mechanism $f_{\mathcal{O}}$.

```
Input: An order 𝒪 over {1,...,n} × {1,...,p}.
1 Broadcast 𝒪 to all agents.
2 for t = 1 to np do
3   | Let (j, i) be the t-th element in 𝒪.
4   | Agent j chooses an available item d_{j,i} from D_i.
5   | Broadcast d_{j,i} to all agents.
6 end
```

For any even number p, given any linear order $\mathcal{K} = j_1 \rhd \cdots \rhd j_n$ over the agents, we define the balanced categorical sequential mechanism *(*balanced mechanism *for short) to be the mechanism with the following order: for any $i \leq p$, the elements between the $((i-1)n+1)$-th position and the $(i \times n)$-th position in $\mathcal{O}$ is* $\begin{cases} (j_1, i) \rhd (j_2, i) \rhd \cdots \rhd (j_n, i) & \text{if } i \text{ is odd} \\ (j_n, i) \rhd (j_{n-1}, i) \rhd \cdots \rhd (j_1, i) & \text{if } i \text{ is even} \end{cases}$. *For example, when $n = 3$, $p = 2$, and $\mathcal{K} = 1 \rhd 2 \rhd 3$, the balanced mechanism uses the order* $(1,1) \rhd (2,1) \rhd (3,1) \rhd (3,2) \rhd (2,2) \rhd (1,2)$.

Categorical sequential mechanisms have the following advantages over direct, centralized mechanisms.

• In terms of overcoming the preference bottleneck, categorical sequential mechanisms have low communication cost. Reporting a linear order over $\mathfrak{D}$ requires $\Theta(n^p p \log n)$ bits for each agent, so the total communication cost is $\Theta(n^{p+1} p \log n)$. For categorical sequential mechanisms, broadcasting $\mathcal{O}$ in Step 1 in Algorithm 1 uses $\Theta(n(np \log np)) = \Theta(n^2 p \log np)$ bits, then in each round broadcasting the choice of the active agent uses $\Theta(n \log n)$ bits. Since there are totally np rounds, the total communication complexity of Algorithm 1 is $\Theta(n^2 p \log n + np(n \log n)) = \Theta(n^2 p \log np)$, which has a $\Theta(n^{p-2} \cdot \frac{\log n}{\log n + \log p})$ multiplicative saving compared to direct mechanisms. In light of this, categorical sequential mechanisms preserve more privacy as well.

• In terms of overcoming the computational bottleneck, categorical sequential mechanisms are easy to compute. Moreover, we feel that agents are more comfortable with choosing an item from a given category per step, compared to reporting a full ranking over all n^p bundles in $\mathfrak{D}$. Hence, categorical sequential mechanisms may also impose a lighter psychological burden to the agents for them to figure our their preferences.

To predict the outcomes of categorical sequential mechanisms, we must make some assumptions about agents' behavior. Since categorical sequential mechanisms are indirect mechanisms, truthfulness is not well-defined, which means that it is not clear what an altruistic agent will do (unlike in direct mechanisms, an altruistic agent naturally reports her true preferences). In this paper, we investigate two types of altruistic and myopic agents. For any l with $1 \leq l \leq p$, we let $D_{l,t}$ denote the set of remaining items in D_l at the beginning of round t. That is, $D_{l,t}$ consists of the items that have not been chosen by any agents in previous rounds.

— Type 1: *optimistic* agents. When an optimistic agent j is active in round t and is about to choose an item from $D_{i,t}$, she will choose the i-th component of her top-ranked bundle that is still available, conditioned on the items she has chosen in previous steps.

— Type 2: *pessimistic* agents. When a pessimistic agent j is active in round t and is about to choose an item from $D_{i,t}$, she will choose an item $d_{j,i}$ such that for all $d'_i \in D_{i,t}$ with $d'_i \neq d_{j,i}$, there exists an available bundle $\vec{d'}$ whose i-th component is d'_i and agent j prefers all available bundles whose i-th component is $d_{j,i}$ to $\vec{d}'$, conditioned on the items she has chosen in previous steps.

Example 3 *Let $n = 3$, $p = 2$. Suppose there are three agents whose preferences are the same as in Example 1, and are simplified as follows, where "others" represents an order over the remaining bundles (and this order will not affect the outcome of the categorical sequential mechanism in this example).*

$$\begin{aligned} \text{Agent 1 (optimistic): } & 12 \succ 21 \succ \text{others} \succ 11 \\ \text{Agent 2 (optimistic): } & 32 \succ \text{others} \succ 22 \\ \text{Agent 3 (pessimistic): } & 13 \succ \text{others} \succ 33 \succ 31 \succ 23 \end{aligned}$$

Let $\mathcal{O} = [(1,1) \rhd (2,2) \rhd (3,1) \rhd (3,2) \rhd (2,1) \rhd (1,2)]$. Suppose agent 1 and agent 2 are optimistic and agent 3 is pessimistic.

When $t = 1$, 12 is the top-ranked available bundle for agent 1. Since agent 1 is optimistic, she chooses 1 from D_1. When $t = 2$, 32 is the top-ranked available bundle for agent 2. Since agent 2 is optimistic, she chooses 2 from D_2. When $t = 3$, the available bundles are $\{2,3\} \times \{1,3\}$. If agent 3 chooses 2 from D_1, then the worst-case available bundle is 23, and if agent 3 chooses 3, then the worst-case available bundle is 31. Since agent 3 prefers 31 to 23, she will choose 3 from D_1. When $t = 4$, the available bundles are $\{3\} \times \{1,3\}$, and agent 3 will choose 3 from D_2. Then, when $t = 5$, agent 2 choses 2 from D_1 and when $t = 6$, agent 1 choses 1 from D_2. The final allocation is: agent 1 gets 11, agent 2 gets 22, and agent 3 gets 33.

We emphasize that in this paper it is assumed that whether an agent is optimistic or pessimistic is determined before the allocation process, and each agent is myopic and stays optimistic (respectively, pessimistic) throughout the allocation process. In the above example, since agent 3 chooses items in $t = 3$ and $t = 4$, if she is strategic and willing to look ahead for one step, then her dominant strategy is to choose the items in her top-ranked available bundle. Analysis of such strategic and self-interested agents is beyond the scope of this paper and is left for future research.

5. ORDINAL EFFICIENCY OF CATEGORICAL SEQUENTIAL ALLOCATION PROTOCOLS

In this section, we focus on the *ordinal efficiency* of categorical sequential mechanisms by evaluating the outcome of an allocation mechanism by the vector composed of individual agents' *ranks* of the bundles allocated to them.[4] Due to the space constraint, we only present proof sketches for Proposition 1 and Theorem 2. Full proofs for all propositions and the theorem can be found in the appendix. Given a categorical sequential mechanism $f_{\mathcal{O}}$, we introduce the following notation for any $j \leq n$.

- Let $\mathcal{O}_j$ denote the linear order over $\{1, \ldots, p\}$ according to which agent j chooses items in $\mathcal{O}$.
- For any $i \leq p$, let $k_{j,i}$ denote the number of items in D_i that have *not* been chosen by other agents before agent j chooses an item from D_i. Formally, $k_{j,i} = 1 + |\{(j', i) : (j,i) \rhd_{\mathcal{O}} (j', i)\}|$. Equivalently $k_{j,i} = n - |\{(j', i) : (j', i) \rhd_{\mathcal{O}} (j,i)\}|$.
- Let K_j denote the smallest natural number such that no agent can "interrupt" agent j from choosing all items in her top-ranked bundle that is available in round $(j, \mathcal{O}_k(j))$. Formally, K_j is the smallest number such that for any l with $K_j < l \leq p$, between the round when agent j chooses an item from category $\mathcal{O}_j(K_j)$ and the round when agent j chooses an item from category $\mathcal{O}_j(l)$, no agent chooses an item from category $\mathcal{O}_j(l)$. We note that K_j is defined solely by $\mathcal{O}$, which means that it does not

[4]We note that this definition is different from the ordinal efficiency for randomized allocation mechanisms [Bogomolnaia and Moulin 2001].

depend on the agents' preferences. Agents' preferences will be considered in the worst-case analysis soon. We also note that K_j represents the category ranked at the K_j-th position in $\mathcal{O}_j$, which is not necessarily the category K_j.

Example 4 *Let $\mathcal{O}^* = [(1,1) \rhd (1,2) \rhd (1,3) \rhd (2,1) \rhd (2,2) \rhd (2,3) \rhd (3,1) \rhd (3,2) \rhd (3,3)]$. That is, $f_{\mathcal{O}^*}$ is a serial dictatorship. Then $\mathcal{O}_1^* = \mathcal{O}_2^* = \mathcal{O}_3^* = 1 \rhd 2 \rhd 3$. $K_1 = K_2 = K_3 = 1$. $k_{1,1} = k_{1,2} = k_{1,3} = 3$, $k_{2,1} = k_{2,2} = k_{2,3} = 2$, $k_{3,1} = k_{3,2} = k_{3,3} = 1$.*

Let $\mathcal{O}$ be the order in Example 3, that is, $\mathcal{O} = [(1,1) \rhd (2,2) \rhd (3,1) \rhd (3,2) \rhd (2,1) \rhd (1,2)]$.

$\mathcal{O}_1 = 1 \rhd 2$. $K_1 = 2$ since $(2,2)$ is between $(1,1)$ and $(1,2)$ in $\mathcal{O}$. $k_{1,1} = 3$, $k_{1,2} = 1$.

$\mathcal{O}_2 = 2 \rhd 1$. $K_2 = 2$ since $(3,1)$ is between $(2,2)$ and $(2,1)$. $k_{2,1} = 1$, $k_{2,2} = 3$.

$\mathcal{O}_3 = 1 \rhd 2$. $K_3 = 1$ since between $(3,1)$ and $(3,2)$ in $\mathcal{O}$, no agent chooses an item from D_2. $k_{3,1} = k_{3,2} = 2$.

Proposition 1 *For any categorical sequential mechanism $f_{\mathcal{O}}$, any combination of optimistic and pessimistic agents, and any $j \leq n$, we have the following upper bounds.*

— **Upper bound for any optimistic agent:** *if j is optimistic, then the rank of the bundle allocated to her is at most $n^p + 1 - \prod_{l=K_j}^{p} k_{j,O_j(l)}$.*

— **Upper bound for any pessimistic agent:** *if j is pessimistic, then the rank of the bundle allocated to her is at most $n^p - \sum_{l=1}^{p}(k_{j,O_j(l)} - 1)$.*

Proof sketch: W.l.o.g. let $\mathcal{O}_j = 1 \rhd 2 \rhd \cdots \rhd p$. If j is optimistic, then we let $t_j = \mathcal{O}^{-1}(j, K_j)$ and let $(d_{j,1}, \ldots, d_{j,K_j-1}) \in D_1 \times \cdots \times D_{K_j-1}$ denote the items agent j chose in the previous rounds. It follows that at the beginning of round t_j, the following $\prod_{l=K_j}^{p} k_{j,l}$ bundles are available for agent j: $\mathfrak{D}_j = (d_{j,1}, \ldots, d_{j,K_j-1}) \times \prod_{l=K_j}^{p} D_{l,t_j}$. By the definition of K_j, no agent can interrupt agent j from choosing the items in her top-ranked bundle in $\mathfrak{D}_j$, and $|\mathfrak{D}_j| = \prod_{l=K_j}^{p} k_{j,l}$.

If j is pessimistic, then we let $\vec{d_j} = (d_{j,1}, \ldots, d_{j,p}) = f_{\mathcal{O}}^j(P)$ denote her allocation by $f_{\mathcal{O}}$. By the definition of pessimism and the assumption that for any $1 \leq l \leq p$, in round $t^* = \mathcal{O}^{-1}(j,l)$ agent j chose $d_{j,l}$ from D_{l,t^*}, we must have that for all $d'_l \in D_{l,t^*}$ with $d'_l \neq d_{j,l}$, there exists an bundle $(d_{j,1}, \ldots, d_{j,l-1}, d'_l, \ldots, d'_p)$ that is ranked below $\vec{d_j}$. Such bundles are all different and the number of them is $\sum_{l=1}^{p}(k_{j,l} - 1)$, which proves the proposition for pessimistic agents. ■

We note that in Proposition 1 applies to any combination of optimistic and pessimistic agents (e.g. in Example 3), which is much more general than the setting with all-optimistic agents and setting with all-pessimistic agents. K_j's are only used to present the upper bounds for optimistic agents, and $k_{j,\mathcal{O}_j(l)}$ for all $l < K_j$ are only used to present the upper bounds for pessimistic agents.

Our main theorem in this section states that for all combinations of optimistic and pessimistic agents, all upper bounds described in Proposition 1 can be matched in one preference profile. Surprisingly, for the same profile there exists an allocation where almost all agents get their top-ranked bundle (and the only agent who may not get her top-ranked bundle gets her second-ranked bundle). Therefore, the theorem is not only a worst-case analysis in the absolute sense (just considering the bounds themselves), but also in the comparative sense (the bounds are compared to the optimal allocation of the same profile, which has almost optimal ordinal efficiency for every agent).

Theorem 2 *For any categorical sequential mechanism $f_{\mathcal{O}}$ and any combination of optimistic and pessimistic agents, there exists a preference profile P such that for all $j \leq n$:*

(*1*) *if agent j is optimistic, then the rank of the bundle allocated to her is $n^p + 1 - \prod_{l=K_j}^{p} k_{j,O_j(l)}$;*

(2) *if agent j is pessimistic, then the rank of the bundle allocated to her is* $n^p - \sum_{l=1}^{p}(k_{j,O_j(l)} - 1)$*;*
(3) *there exists an allocation where at least* $n-1$ *agents get their top-ranked bundles, and the remaining agent gets her top-ranked or second-ranked bundle. Moreover, if the first agent in* $\mathcal{O}$ *is pessimistic, then there exists an allocation where all agents get their top-ranked bundle.*

Proof sketch: We prove the theorem by constructing a preference profile P such that in $f_{\mathcal{O}}(P)$, for all $j \leq n$, agent j gets $(j, \ldots, j)$. For all i and t, define $D^*_{i,t}$ to be the subset of $D_i = \{1, \ldots, n\}$ such that $q \in D^*_{i,t}$ if and only if agent q has not chosen an item from D_i before the t-th round. By definition, if $\mathcal{O}(t) = (j, i)$ then $j \in D^*_{i,t}$. Formally, $D^*_{i,t} = \{q \leq n : \mathcal{O}^{-1}(q, i) \geq t\}$. Then, we let $t^*_{j,l} = \mathcal{O}^{-1}(j, \mathcal{O}_j(l))$ denote the round when j chooses from category $\mathcal{O}_j(l)$. Let $\mathcal{O}(1) = (j_1, i_1)$. That is, agent j_1 is the first agent to choose an item in $f_{\mathcal{O}}$, and she chooses from category D_{i_1}. Let L_{i_1} denote the order over $\{1, \ldots, n\}$ representing the order for the *agents* to choose items from D_{i_1} in $\mathcal{O}$. That is, $j \rhd_{L_{i_1}} j'$ if and only if $(j, i_1) \rhd_{\mathcal{O}} (j', i_1)$. By definition we have $j_1 = L_{i_1}(1)$. For any $j \leq n$, we let $Pred_{i_1}(j) = L_{i_1}(L^{-1}_{i_1}(j) - 1)$ denote the predecessor of agent j in L_{i_1}, that is, the latest agent who chose an item from category i_1 before agent j chooses from category i_1. If $j = 1$, then we let the last agent in L_{i_1} be her predecessor, that is, $Pred_{i_1}(1) = L_{i_1}(n)$.

If agent j is optimistic, then we let the following bundles be ranked in the bottom of her preferences: $\text{BottomBundles}_j^{\text{Opt}} = (j_{\mathcal{O}_j(1)}, \ldots, j_{\mathcal{O}_j(K_j-1)}) \times \prod_{l=K_j}^{p} D^*_{\mathcal{O}_j(l), t^*_{j,l}}$, and let agent j's preferences be any linear order compatible with the partial order specified in Table II.

Table II. Partial preferences for an optimistic agent j.

Optimistic agent		Order
$j \neq j_1$	case 1: $K_j = 1$	$([Pred_{i_1}(j)]_{i_1}, [j]_{-i_1}) \succ \cdots \succ (j, \ldots, j) \succ$ others in $\text{BottomBundles}_j^{\text{Opt}}$
	case 2: $K_j > 1$	$([Pred_{i_1}(j)]_{i_1}, [j]_{-i_1}) \succ ([Pred_{\mathcal{O}_j(K_j)}(j)]_{\mathcal{O}_j(K_j)}, [j]_{-\mathcal{O}_j(K_j)})$ $\succ \cdots \succ (j, \ldots, j) \succ$ others in $\text{BottomBundles}_j^{\text{Opt}}$
$j = j_1$	case 1: $K_j = 1$	$(j_1, \ldots, j_1) \succ ([L_{i_1}(n)]_{i_1}, [j_1]_{-i_1}) \succ$ others
	case 2: $K_j > 1$	$([Pred_{\mathcal{O}_j(K_j)}(j_1)]_{\mathcal{O}_j(K_j)}, [j_1]_{-\mathcal{O}_j(K_j)}) \succ ([L_{i_1}(n)]_{i_1}, [j_1]_{-i_1})$ $\succ \cdots \succ (j_1, \ldots, j_1) \succ$ others in $\text{BottomBundles}_{j_1}^{\text{Opt}}$

If agent j is pessimistic, then we first define the following bundles: $\text{BottomBundles}_j^{\text{Pes}} = \bigcup_{l=1}^{p} \bigcup_{d \in D^*_{\mathcal{O}_j(l), t^*_{j,l}}} \{([d]_{\mathcal{O}_j(l)}, [j]_{-\mathcal{O}_j(l)})\}$, where $[j]_{-\mathcal{O}_j(l)}$ means that all components except the $\mathcal{O}_j(l)$-th component is j. Bundles in $\text{BottomBundles}_j^{\text{Pes}}$ are (partially) ranked as follows: first, $(j, \ldots, j)$ is ranked on the top of them; then, for any $1 \leq l_1 < l_2 \leq p$, any $d_1 \in D^*_{\mathcal{O}_j(l_1), t^*_{j,l_1}}$ with $d_1 \neq j$, and any $d_2 \in D^*_{\mathcal{O}_j(l_2), t^*_{j,l_2}}$ with $d_2 \neq j$, we rank $([d_1]_{\mathcal{O}_j(l_1)}, [j]_{-\mathcal{O}_j(l_1)})$ below $([d_2]_{\mathcal{O}_j(l_2)}, [j]_{-\mathcal{O}_j(l_2)})$. Then, we let agent j's preferences be any linear order compatible of the partial order specified in Table III.

Table III. Partial preferences for a pessimistic agent j.

Pessimistic agent	Order
$j \neq j_1$	$([Pred_{i_1}(j)]_{i_1}, [j]_{-i_1}) \succ \cdots \succ (j, \ldots, j) \succ$ others in $\text{BottomBundles}_j^{\text{Pes}}$
$j = j_1$	$([L_{i_1}(n)]_{i_1}, [j_1]_{-i_1}) \succ \cdots \succ (j_1, \ldots, j_1) \succ$ others in $\text{BottomBundles}_{j_1}^{\text{Pes}}$ $\succ (L_{i_1}(n), \ldots, L_{i_1}(n))$

For the constructed preference profile, we then prove by induction on the round in the execution of $f_{\mathcal{O}}$ that the active (optimistic or pessimistic) agent j always chooses item j from the designated category. This proves part (1) and part (2) of the theorem. For part (3), we consider the allocation where agent j gets $([Pred_{i_1}(j)]_{i_1}, [j]_{-i_1})$. In this allocation, all agents except j_1 get their top-ranked bundle, and j_1 gets her top-ranked bundle (if j_1 is pessimistic) or second-ranked bundle (if j_1 is optimistic). ■

Example 5 *Let $\mathcal{O}$ be the order in Example 3. Suppose agent 1 and 2 are optimistic and agent 3 is pessimistic. By Theorem 2 and Example 4, there exists a profile P such that after applying $f_{\mathcal{O}}$, the bundle allocated to agent 1 is ranked at the last position ($K_1 = 2$ and $k_{1,2} = 1$); the bundle allocated to agent 2 is ranked at the last position ($K_2 = 2$ and $k_{2,1} = 1$); and the bundle allocated to agent 3 is ranked at the 3rd position from the bottom ($k_{3,1} = k_{3,2} = 2$). Moreover, there exists an allocation where agent 2 and 3 get their top-ranked bundles and agent 1 gets her second-ranked bundle. In fact, the preference profile described in Example 3 satisfies all these conditions.*

We emphasize that in Theorem 2, whether an agent is optimistic or pessimistic is fixed before we know their preferences. The theorem provides a useful tool to compare various categorical sequential mechanisms with optimistic and pessimistic agents in the utilitarian sense and egalitarian sense, as we will show in the propositions in the rest of this section. We recall that for any allocation mechanism f, any $j \leq n$, and any preference profile P, $f^j(P)$ is the bundle allocated to agent j. For any linear order R over $\mathfrak{D}$ and any bundle $\vec{d}$, we let $\text{Rank}(R, \vec{d})$ denote the *rank* of $\vec{d}$ in R, where the highest position has rank 1 and the lowest position has rank n^p.

Definition 3 *Given any categorical sequential mechanism $f_{\mathcal{O}}$, any n, and any combination of optimistic and pessimistic agents, we let the* worst-case utilitarian rank *be* $\max_{P_n} \sum_{R_j \in P_n} \textit{Rank}(R_j, f^j_{\mathcal{O}}(P_n))$, *and the* worst-case egalitarian rank *be* $\max_{P_n} \max_{R_j \in P_n} \textit{Rank}(R_j, f^j_{\mathcal{O}}(P_n))$, *where P_n is a preference profile of n agents.*

In words, the worst-case utilitarian rank is the worst (largest) total rank of the allocation by $f_{\mathcal{O}}$ w.r.t. agents' preferences. The worst-case egalitarian rank is the worst (largest) rank of the least-satisfied agent. In both notions the worst-case is taken over all preference profiles of n agents. The negation of utilitarian rank naturally corresponds to *utilitarian social welfare* w.r.t. the *Borda utility function*, where for all $l \leq n^p$, an agent's utility for the l-th ranked alternative is $n^p - l$. Similarly, the negation of egalitarian rank naturally corresponds to *egalitarian social welfare*.

Proposition 2 *Among all categorical sequential mechanisms, serial dictatorships with all-optimistic agents have the best (smallest) worst-case utilitarian rank and the worst (largest) worst-case egalitarian rank.*

Proposition 3 *Any categorical sequential mechanisms with all-optimistic agents has the worst (largest) worst-case egalitarian rank, which is n^p.*

Proposition 4 *For any even number p, the worst-case egalitarian rank of any balanced mechanism with all-pessimistic agents (see Example 2) is $n^p - (n-1)p/2$. These are the mechanisms with the best worst-case egalitarian rank among categorical sequential mechanisms with all-pessimistic agents.*

A natural question after Proposition 4 is: when p is even, are the balanced mechanisms with all-pessimistic agents optimal in terms of worst-case egalitarian rank, among all categorical sequential mechanisms *for any combination of optimistic and pessimistic agents*? The answer is negative due to the following proposition.

Proposition 5 *For any even number p with $2^p > 1+(n-1)p/2$, there exists a categorical sequential mechanism with both optimistic and pessimistic agents, whose worst-case egalitarian rank is strictly better (smaller) than $n^p - (n-1)p/2$.*

Concluding remarks for Section 5: Our main theorem in this section (Theorem 2) and the following propositions shed some light on applications of categorical sequential mechanisms for altruistic agents. For example, Proposition 2 tells us that if our goal is to achieve good worst-case utilitarian rank (i.e. optimal worst-case utilitarian social welfare w.r.t. the Borda utility function), then the optimal categorical sequential mechanisms are serial dictatorships, and in this case we should advise the altruistic agents to be optimistic. However, all categorical sequential mechanisms with all-optimistic agents have the worst worst-case egalitarian rank (Proposition 3), which means that we should avoid advising all agents to be optimistic if worst-case egalitarian rank is our concern. If we want to achieve good worst-case egalitarian rank, then the balanced categorical sequential mechanisms are much better options, for which we can advise the altruistic agents to be pessimistic (Proposition 4). And it is possible to achieve even better worst-case egalitarian rank by a categorical sequential mechanism where some agents are optimistic and others are pessimistic, depending on their positions in the order $\mathcal{O}$ (Proposition 5).

6. SIMULATION RESULTS

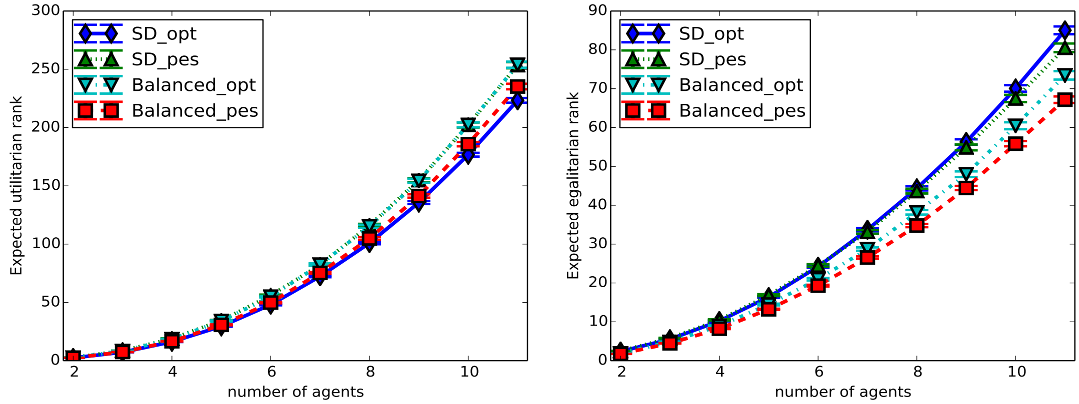
In this section, we use computer simulations to evaluate *expected* efficiency of categorical sequential mechanisms, when agents' preferences are generated i.i.d. from a well-known statistical model called the *Mallows model* [Mallows 1957]. Similarly to the worst-case analysis in the previous section, we evaluate two types of expected ranks: the expected utilitarian rank and the expected egalitarian rank. We first recall the definition of the Mallows model.

Definition 4 *Let $\mathcal{C}$ denote a set of alternatives and let $\mathcal{L}(\mathcal{C})$ denote the set of all linear orders over $\mathcal{C}$. In a* Mallows model*, each parameter consists of a ground truth linear order W over $\mathcal{C}$ and a* dispersion parameter $0 < \varphi \leq 1$. *Given (W, φ), the probability to generate a linear order V over $\mathcal{C}$ is $\Pr(V|W,\varphi) = \frac{1}{Z}\cdot\varphi^{\mathit{Kendall}(V,W)}$, where Kendall$(V,W)$ is the* Kendall-tau distance *between V and W, defined to be the number of different pairwise comparisons between alternatives. $Z = \sum_{V\in\mathcal{L}(\mathcal{C})}\varphi^{\mathit{Kendall}(V,W)}$ is the normalization factor.*

In the Mallows model, the dispersion parameter measures the centrality of the generated linear orders. The smaller φ is, the more centralized the randomly generated linear orders are (around the ground truth linear order). When $\varphi = 1$, the Mallows model degenerates to the uniform distribution for any ground truth linear order W.

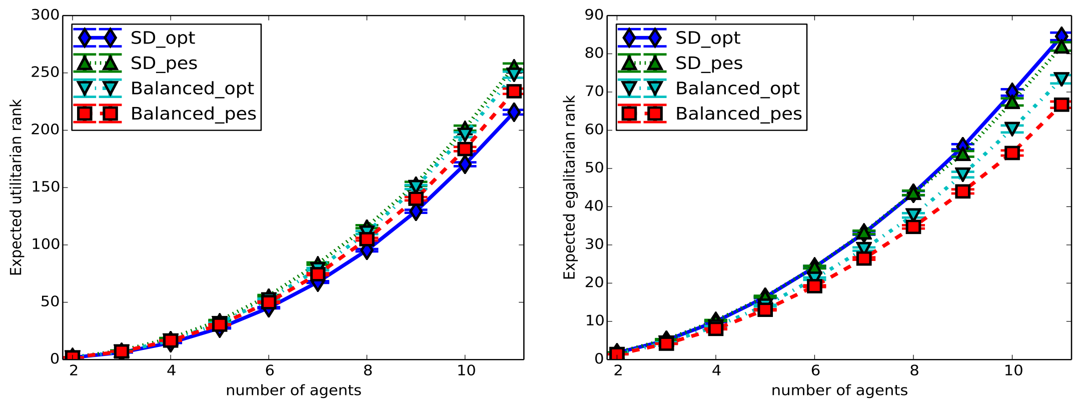
Data generation. In our experiments, we fix $p = 2$, let n range from 2 to 11, and let φ be 0.1, 0.5, and 1. For each setting, we first randomly generate a linear order W over $\mathfrak{D}$, and then use it as the ground truth linear order in the Mallows model to generate n agents' preferences. For each setting we generate 2000 datasets and use them to approximately compute the expected utilitarian rank and the expected egalitarian rank, defined by replacing $\max_{P_n}$ by E_{P_n} in Definition 3.[5] We evaluate serial dictatorships and balanced mechanisms with two configurations of agents: all-optimistic agents and all-pessimistic agents. All computations were done on a 1.8 GHz Intel Core i7 laptop with 4GB memory.

[5]The expected egalitarian rank should be distinguished from the *egalitarian expected rank*, which first computes the expected rank for every agent, then chooses the largest (expected) rank.



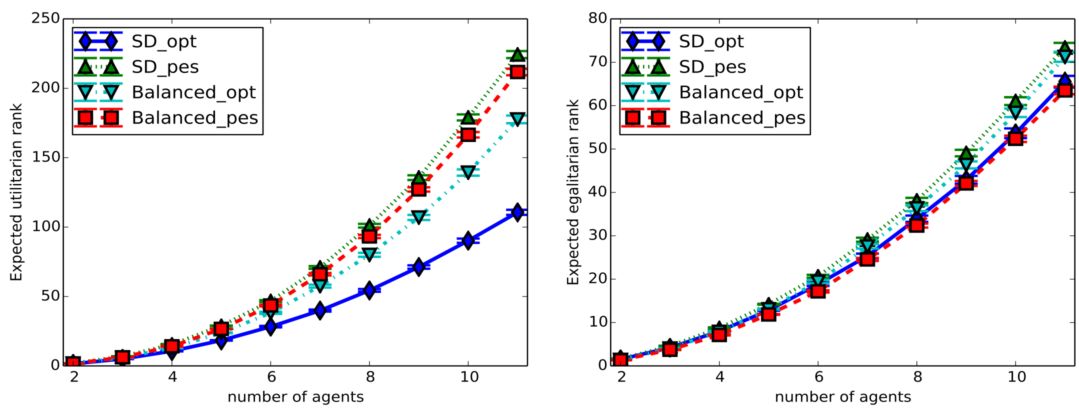
Expected utilitarian rank. Expected egalitarian rank.

Fig. 1. The data are generated from the Mallows model with $\varphi = 0.1$.



Expected utilitarian rank. Expected egalitarian rank.

Fig. 2. The data are generated from the Mallows model with $\varphi = 0.5$.



Expected utilitarian rank. Expected egalitarian rank.

Fig. 3. The data are generated from the uniform distribution (the Mallows model with $\varphi = 1$).

Results. Our results are summarized in Figure 1, 2, and 3. In each figure we also plot 95% confidence intervals. It can be seen from the figures that in general, serial dictatorships with all-optimistic agents have the best (smallest) expected utilitarian rank, and balanced mechanisms with all-pessimistic agents have the best (smallest) expected egalitarian rank. All these comparisons are statistically significant at the 0.05 level, except for the case of expected egalitarian rank when $\varphi = 1$ (namely, the uniform distribution), where the performance of serial dictatorships with all-optimistic agents and the performance of the balanced mechanisms with all-pessimistic agents are too close to draw informative statistical conclusions. These observations complement and are (incidentally) consistent with the worst-case results obtained in the previous section, which tell us that among the four types of mechanisms, serial dictatorships with all-optimistic agents have the best *worst-case* utilitarian rank, and the balanced mechanisms with all-pessimistic agents have the best *worst-case* egalitarian rank.

7. FUTURE WORK

There are many immediate open questions, including analyzing the outcomes and ordinal efficiency for categorical sequential mechanisms for other types of agents, including strategic and self-interested agents, and minimax-regret agents. We also plan to work on theoretical analysis of expected utilitarian rank and egalitarian rank, and randomized allocation mechanisms.

More excitingly, we feel that CDAPs provide a natural framework for applying techniques in many other fields to overcome the preference bottleneck, computational bottleneck, and threats of agents' strategic behavior discussed in the Introduction. For example, we plan to design preference representation languages and mechanisms for general CDAPs, including natural generalizations of CP-nets [Boutilier et al. 2004], LP-trees [Booth et al. 2010], and soft constraints [Pozza et al. 2011]. We also plan to evaluate new mechanisms w.r.t. fairness and computational resistance to various kinds of malicious behavior, including manipulation, bribery, and control. In this paper, we implicitly assumed that all items are non-sharable, meaning that each item can only be exclusively allocated to one agent. Designing mechanisms for CDAPs with both sharable and non-sharable items is also a natural and promising next step.

Acknowledgements

Lirong Xia is supported by an RPI startup fund. We thank Vincent Conitzer, Jerome Lang, David Parkes, and participants of ISAIM-14 special session on Computational Social Choice for useful feedbacks. We thank Jonathan Huang for allowing us to use his codes to generate data from the Mallows model.

REFERENCES

BOGOMOLNAIA, A. AND MOULIN, H. 2001. A New Solution to the Random Assignment Problem. *Journal of Economic Theory 100,* 2, 295–328.

BOOTH, R., CHEVALEYRE, Y., LANG, J., MENGIN, J., AND SOMBATTHEERA, C. 2010. Learning conditionally lexicographic preference relations. In *Proceeding of the 2010 conference on ECAI 2010: 19th European Conference on Artificial Intelligence*. Amsterdam, The Netherlands, 269–274.

BOUTILIER, C., BRAFMAN, R., DOMSHLAK, C., HOOS, H., AND POOLE, D. 2004. CP-nets: A tool for representing and reasoning with conditional ceteris paribus statements. *Journal of Artificial Intelligence Research 21*, 135–191.

BOUTILIER, C., CARAGIANNIS, I., HABER, S., LU, T., PROCACCIA, A. D., AND SHEFFET, O. 2012. Optimal social choice functions: A utilitarian view. In *ACM Conference on Electronic Commerce*. Valencia, Spain, 197–214.

BOUVERET, S. AND LANG, J. 2011. A general elicitation-free protocol for allocating indivisible goods. In *Proceedings of the Twenty-Second International Joint Conference on Artificial Intelligence (IJCAI)*. Barcelona, Catalonia, Spain, 73–78.

BRAMS, S. J., JONES, M. A., AND KLAMLER, C. 2006. Better Ways to Cut a Cake. *Notices of the AMS 53,* 11, 1314–1321.

BRAMS, S. J., KILGOUR, D. M., AND ZWICKER, W. S. 1998. The paradox of multiple elections. *Social Choice and Welfare 15,* 2, 211–236.

BRANDT, F., CONITZER, V., AND ENDRISS, U. 2013. Computational social choice. In *Multiagent Systems*, G. Weiss, Ed. MIT Press.

BUDISH, E. AND CANTILLON, E. 2012. The Multi-Unit Assignment Problem: Theory and Evidence from Course Allocation at Harvard. *American Economic Review 102,* 5, 2237–71.

CHEVALEYRE, Y., DUNNE, P. E., ENDRISS, U., LANG, J., LEMAITRE, M., MAUDET, N., PADGET, J., PHELPS, S., RODRÍGUEZ-AGUILAR, J. A., AND SOUSA, P. 2006. Issues in multiagent resource allocation. *Informatica 30*, 3–31.

EHLERS, L. AND KLAUS, B. 2003. Coalitional strategy-proof and resource-monotonic solutions for multiple assignment problems. *Social Choice Welfare 21*, 265—280.

HATFIELD, J. W. 2009. Strategy-proof, efficient, and nonbossy quota allocations. *Social Choice and Welfare 33,* 3, 505–515.

KOUTSOUPIAS, E. AND PAPADIMITRIOU, C. 1999. Worst-case Equilibria. In *Proceedings of the 16th Annual Conference on Theoretical Aspects of Computer Science*. Trier, Germany, 404–413.

LACY, D. AND NIOU, E. M. 2000. A problem with referendums. *Journal of Theoretical Politics 12,* 1, 5–31.

MALLOWS, C. L. 1957. Non-null ranking model. *Biometrika 44,* 1/2, 114–130.

PAPADIMITRIOU, C. H. AND STEIGLITZ, K. 1998. *Combinatorial Optimization: Algorithms and Complexity*. Dover Publications.

PÁPAI, S. 2000a. Strategyproof assignment by hierarchical exchange. *Econometrica 68,* 6, 1403–1433.

PÁPAI, S. 2000b. Strategyproof multiple assignment using quotas. *Review of Economic Design 5*, 91–105.

PÁPAI, S. 2001. Strategyproof and nonbossy multiple assignments. *Journal of Public Economic Theory 3,* 3, 257–71.

POZZA, G. D., PINI, M. S., ROSSI, F., AND VENABLE, K. B. 2011. Multi-agent soft constraint aggregation via sequential voting. In *Proceedings of the Twenty-Second International Joint Conference on Artificial Intelligence*. Barcelona, Catalonia, Spain, 172–177.

PROCACCIA, A. D. AND ROSENSCHEIN, J. S. 2006. The Distortion of Cardinal Preferences in Voting. In *Proceedings of the 10th International Workshop on Cooperative Information Agents*. LNAI Series, vol. 4149. 317–331.

SÖNMEZ, T. AND ÜNVER, M. U. 2011. Matching, Allocation, and Exchange of Discrete Resources. In *Handbook of Social Economics*, J. Benhabib, A. Bisin, and M. O. Jackson, Eds. North-Holland, Chapter 17, 781–852.

SVENSSON, L.-G. 1999. Strategy-proof allocation of indivisible goods. *Social Choice and Welfare 16,* 4, 557–567.

Appendix: Full Proofs

Lemma 1. *Let f be a strategy-proof and non-bossy allocation mechanism over a basic categorized domain with $p \geq 2$. For any pair of preference profiles P and P' such that for all $j \leq n$, $\{\vec{d} \in \mathfrak{D} : \vec{d} \succ_{R'_j} f^j(P)\} \subseteq \{\vec{d} \in \mathfrak{D} : \vec{d} \succ_{R_j} f^j(P)\}$, we have $f(P') = f(P)$.*

Proof: We first prove the lemma for the special case where P and P' only differ on one agent's preferences. Let j be an agent with $R'_j \neq R_j$ and $\{\vec{d} \in \mathfrak{D} : \vec{d} \succ_{R'_j} f^j(P)\} \subseteq \{\vec{d} \in \mathfrak{D} : \vec{d} \succ_{R_j} f^j(P)\}$. We will prove that $f^j(R'_j, R_{-j}) = f^j(R_j, R_{-j})$.

Suppose for the sake of contradiction $f^j(R'_j, R_{-j}) \neq f^j(R_j, R_{-j})$. If $f^j(R'_j, R_{-j}) \succ_{R_j} f^j(R_j, R_{-j})$ then it means that f is not strategy-proof since j has incentive to report R'_j when her true preferences are R_j. If $f^j(R_j, R_{-j}) \succ_{R_j} f^j(R'_j, R_{-j})$ then $f^j(R_j, R_{-j}) \succ_{R'_j} f^j(R'_j, R_{-j})$, which means that when agent j's preferences are R'_j she has incentive to report R_j,. This again contradicts the assumption that f is strategy-proof. Therefore $f^j(R_j, R_{-j}) = f^j(R'_j, R_{-j})$.

By non-bossiness, $f(R_j, R_{-j}) = f(R'_j, R_{-j})$. The lemma is proved by recursively applying this argument to $j = 1, \ldots, n$. ■

Lemma 2. *Let f be a strategy-proof and non-bossy allocation mechanism over a basic categorized domain with $p \geq 2$. For any profile P, any $j \leq n$, any bundle $\vec{d}$, and any R'_j that is a pushup of $\vec{d}$ from R_j, either (1) $f(R'_j, R_{-j}) = f(R)$ or (2) $f^j(R'_j, R_{-j}) = \vec{d}$.*

Proof: We first prove that $f^j(R'_j, R_{-j}) = f^j(R)$ or $f^j(R'_j, R_{-j}) = \vec{d}$. Suppose on the contrary that $f^j(R'_j, R_{-j})$ is neither $f^j(R)$ nor $\vec{d}$. If $f^j(R'_j, R_{-j}) \succ_{R_j} f^j(R)$, then f is not strategy-proof since when agent j's true preferences are R_j and other agents' preferences are R_{-j}, she has incentive to report R'_j to make her allocation better. If $f^j(R) \succ_{R_j} f^j(R'_j, R_{-j})$, then since $\vec{d} \neq f^j(R'_j, R_{-j})$, we have $f^j(R) \succ_{R'_j} f^j(R'_j, R_{-j})$. In this case when agent j's true preferences are R'_j and other agents' preferences are R_{-j}, she has incentive to report R_j to make her allocation better, which means that f is not strategy-proof. Therefore, $f^j(R'_j, R_{-j}) = f^j(R)$ or $f^j(R'_j, R_{-j}) = \vec{d}$. If $f^j(R'_j, R_{-j}) = f^j(R_j, R_{-j})$, then by non-bossiness $f(R'_j, R_{-j}) = f(R)$. This completes the proof. ■

Lemma 3. *For any basic categorized domains with $p \geq 2$, any strategy-proof, non-bossy, and category-wise neutral allocation mechanism is Pareto optimal.*

Proof: We prove the lemma by contradiction. Let f be a strategy-proof, non-bossy, category-wise neutral, but non-(Pareto optimal) allocation mechanism. Let $P = (R_1, \ldots, R_n)$ denote a profile such that $f(P)$ is Pareto dominated by an allocation A. For any $i \leq m$, let M_i denote the permutation over D_i so that for every $j \leq n$, $[f^j(P)]_i$ is permuted to $[A(j)]_i$. Let $M = (M_1, \ldots, M_m)$. It follows that for all $j \leq n$, $M(f^j(P)) = A(j)$.

Let R'_j denote an arbitrary ranking where $A(j)$ is ranked at the top place, and $f^j(P)$ is ranked at the second place if it is different from $A(j)$. Let R^*_j denote an arbitrary ranking where $f^j(P)$ is ranked at the top place, and $A(j)$ is ranked at the second place if it is different from $f^j(P)$. Let $P' = (R'_1, \ldots, R'_n)$ and $P^* = (R^*_1, \ldots, R^*_n)$. P' and P^* are illustrated as follows.

$$P' = \left\{ \begin{array}{c} R'_1 : A^1 \succ f^1(P) \succ \text{Others} \\ \vdots \\ R'_n : A^n \succ f^n(P) \succ \text{Others} \end{array} \right\}, P^* = \left\{ \begin{array}{c} R^*_1 : f^1(P) \succ A^1 \succ \text{Others} \\ \vdots \\ R^*_n : f^n(P) \succ A^n \succ \text{Others} \end{array} \right\}$$

Since A Pareto dominates $f(P)$, by Lemma 1 we have $f(P') = f(P)$, because for any $j \leq n$, in R'_j the only bundle ranked ahead of $f^j(P)$ is $A(j)$, if it is different from $f^j(P)$, and $A(j)$ is also ranked ahead of $f^j(P)$ in R_j. By Lemma 1 again we have $f(P^*) = f(P)$. Comparing $M(P')$ and P^*, we observe that the only differences are the orderings among $\mathfrak{D}\backslash\{A(j), f^j(P)\}$. Applying Lemma 1 to P^* and $M(P')$, we have that $f(M(P')) = f(P^*) = f(P)$. However, by category-wise neutrality $f(M(P')) = M(f(P')) = A$, which is a contradiction. ∎

Proposition 1. *For any categorical sequential mechanism $f_{\mathcal{O}}$, any combination of optimistic and pessimistic agents, and any $j \leq n$, we have the following upper bounds.*

— **Upper bound for any optimistic agent:** *if j is optimistic, then the rank of the bundle allocated to her is at most $n^p + 1 - \prod_{l=K_j}^{p} k_{j,O_j(l)}$.*
— **Upper bound for any pessimistic agent:** *if j is pessimistic, then the rank of the bundle allocated to her is at most $n^p - \sum_{l=1}^{p}(k_{j,O_j(l)} - 1)$.*

Proof: Equivalently, we need to prove that for any optimistic agent, the bundle allocated to her is ranked no lower than the $(\prod_{l=K_j}^{p} k_{j,O_j(l)})$-th position from the bottom, and for any pessimistic agent, the bundle allocated to her is ranked no lower than the $(1 + \sum_{l=1}^{p}(k_{j,O_j(l)} - 1))$-th position from the bottom.

W.l.o.g. let $\mathcal{O}_j = 1 \rhd 2 \rhd \cdots \rhd p$. That is, agent j chooses items from categories $1, \ldots, p$ in sequence in the sequential allocation. This means that in this proof, for any $l \leq p$, $\mathcal{O}_j(l) = l$. We first prove the proposition for an optimistic agent j. In the beginning of round $t_j = \mathcal{O}^{-1}(j, K_j)$ in Algorithm 1, agent j has already chosen items from $D_1, \ldots, D_{K_j-1}$, and is ready to choose an item from D_{K_j}. We recall that $D_{l,t}$ is the set of remaining items in D_l at the beginning of round t. By definition, $k_{j,l} = |D_{l,t_j}|$. Let $(d_{j,1}, \ldots, d_{j,K_j-1}) \in D_1 \times \cdots \times D_{K_j-1}$ denote the items agent j has chosen in previous rounds. It follows that at the beginning of the round t_j, the following $\prod_{l=K_j}^{p} k_{j,l}$ bundles are available for agent j:

$$\mathfrak{D}_j = (d_{j,1}, \ldots, d_{j,K_j-1}) \times \prod_{l=K_j}^{p} D_{l,t_j}$$

We now show that an optimistic agent j is guaranteed to obtain her top-ranked bundle in $\mathfrak{D}_j$. Intuitively this holds because by the definition of K_j, for any $l \geq K_j$, when it is agent j's round to choose an item from D_l, the l-th component of her top-ranked bundle in $\mathfrak{D}_j$ is always available. Formally, let $\vec{d_j} = (d_{j,1}, \ldots, d_{j,p})$ denote agent j's top-ranked bundle in $\mathfrak{D}_j$. We prove that agent j will choose $d_{j,l}$ from D_l in round $\mathcal{O}^{-1}(j, l)$ by induction on l. The base case $l = K_j$ is straightforward. Suppose she has chosen $d_{K_j}, d_{K_j+1}, \ldots, d_{l'}$ for some $l' \geq K_j$. Then in round $\mathcal{O}^{-1}(j, l'+1)$ when agent j is about to choose an item from $D_{l'+1}$, the following bundles are available:

$$(d_{j,1}, \ldots, d_{j,l'}) \times \prod_{l=l'+1}^{p} D_{l,t_j}$$

This is because by the induction hypothesis, $(d_{j,1}, \ldots, d_{j,l'})$ have been chosen by agent j in previous rounds. Then, by the definition of K_j, for any $l \geq l' + 1$ no agent choses an item from D_l between round $t_j = \mathcal{O}^{-1}(j, K_j)$ and round $\mathcal{O}^{-1}(j, l')$. Hence the remaining items in D_l is still the same as that in round t_j. This means that $\vec{d_j} \in (d_{j,1}, \ldots, d_{j,l'}) \times \prod_{l=l'+1}^{p} D_{l,t_j} \subseteq \mathfrak{D}_j$. Therefore, $\vec{d_j}$ is still agent j's top-ranked available bundle in the beginning of round $\mathcal{O}^{-1}(j, l')$, when she is about to choose an item from $D_{l'+1}$. Hence agent j will choose $d_{j,l'+1}$. This proves the claim for $l = l'+1$, which means that it holds for all $l \leq p$. Therefore, agent j is allocated $\vec{d_j}$ by the sequential allocation protocol. We note that $|\mathfrak{D}_j| = \prod_{l=K_j}^{p} k_{j,l}$. This proves the proposition for optimistic agents.

We next prove the proposition for an pessimistic agent j. Let $\vec{d_j} = (d_{j,1}, \ldots, d_{j,p})$ denote her allocation by the sequential allocation protocol. Since agent j is pessimistic, for any $1 \leq l \leq p$, in round $t^* = \mathcal{O}^{-1}(j, l)$ agent j chose $d_{j,l}$ from D_{l,t^*}, we must have that for all $d'_l \in D_{l,t^*}$ with $d'_l \neq d_{j,l}$, there exists an bundle $(d_{j,1}, \ldots, d_{j,l-1}, d'_l, \ldots, d'_p)$ that is ranked below $\vec{d_j}$. These bundles are all different and the number of all such bundles is $\sum_{l=1}^{p}(k_{j,l} - 1)$, which proves the proposition for pessimistic agents. ■

Theorem 2. *For any categorical sequential mechanism $f_{\mathcal{O}}$ and any combination of optimistic and pessimistic agents, there exists a preference profile P such that for all $j \leq n$:*

(*1*) *if agent j is optimistic, then the rank of the bundle allocated to her is $n^p + 1 - \prod_{l=K_j}^{p} k_{j,O_j(l)}$;*
(*2*) *if agent j is pessimistic, then the rank of the bundle allocated to her is $n^p - \sum_{l=1}^{p}(k_{j,O_j(l)} - 1)$;*
(*3*) *there exists an allocation where at least $n-1$ agents get their top-ranked bundles, and the remaining agent gets her top-ranked or second-ranked bundle. Moreover, if the first agent in $\mathcal{O}$ is pessimistic, then there exists an allocation where all agents get their top-ranked bundle.*

Proof: Given $\mathcal{O}$ and the information on whether each agent j is optimistic or pessimistic, we will construct a preference profile P such that in $\mathcal{O}(P)$, for all $j \leq n$, agent j obtains $(j, \ldots, j)$.

We prove the theorem in the following three steps: in **Step 1: define bottom bundles**, we specify a set of bundles that are ranked in the bottom positions for each agent j, and require $(j, \ldots, j)$ to be ranked on the top of them. In **Step 2: define top bundles**, we specify top-1 and sometimes also top-2 bundles for each agent. Finally in **Step 3: extend to full preference profile**, we take any preference profile that extends the partial orders constructed in the first two steps, and then show that it satisfies all three properties in the theorem. The construction is summarized in Table IV (for optimistic agents) and Table V (for pessimistic agents), which are replications of Table II and Table III, respectively.

Table IV. Partial preferences for an optimistic agent j. BottomBundles$_j^{\text{Opt}}$ is defined in (1). "Others in BottomBundles$_j^{\text{Opt}}$" refers to [BottomBundles$_j^{\text{Opt}} \setminus \{(j, \ldots, j)\}$].

Optimistic agent		Order
$j \neq j_1$	case 1: $K_j = 1$	$([Pred_{i_1}(j)]_{i_1}, [j]_{-i_1}) \succ \cdots \succ (j, \ldots, j) \succ$ others in BottomBundles$_j^{\text{Opt}}$
	case 2: $K_j > 1$	$([Pred_{i_1}(j)]_{i_1}, [j]_{-i_1}) \succ ([Pred_{\mathcal{O}_j(K_j)}(j)]_{\mathcal{O}_j(K_j)}, [j]_{-\mathcal{O}_j(K_j)}) \succ \cdots \succ (j, \ldots, j) \succ$ others in BottomBundles$_j^{\text{Opt}}$
$j = j_1$	case 1: $K_j = 1$	$(j_1, \ldots, j_1) \succ ([L_{i_1}(n)]_{i_1}, [j_1]_{-i_1}) \succ$ others
	case 2: $K_j > 1$	$([Pred_{\mathcal{O}_j(K_j)}(j_1)]_{\mathcal{O}_j(K_j)}, [j_1]_{-\mathcal{O}_j(K_j)}) \succ ([L_{i_1}(n)]_{i_1}, [j_1]_{-i_1}) \succ \cdots \succ (j_1, \ldots, j_1) \succ$ others in BottomBundles$_{j_1}^{\text{Opt}}$

Table V. Partial preferences for a pessimistic agent j. $\text{BottomBundles}_j^{\text{Pes}}$ is defined in (2). For $j \neq j_1$, "others in $\text{BottomBundles}_j^{\text{Pes}}$" refers to $(\text{BottomBundles}_{j_1}^{\text{Pes}} \setminus \{(j,\ldots,j)\})$. For $j = j_1$, "others in $\text{BottomBundles}_{j_1}^{\text{Pes}}$" refers to $(\text{BottomBundles}_{j_1}^{\text{Pes}} \setminus \{(j_1,\ldots,j_1),([L_{i_1}(n)]_{i_1},[j_1]_{-i_1})\})$.

Pessimistic agent	Order
$j \neq j_1$	$([Pred_{i_1}(j)]_{i_1},[j]_{-i_1}) \succ \cdots \succ (j,\ldots,j) \succ$ others in $\text{BottomBundles}_j^{\text{Pes}}$
$j = j_1$	$([L_{i_1}(n)]_{i_1},[j_1]_{-i_1}) \succ \cdots \succ (j_1,\ldots,j_1) \succ$ others in $\text{BottomBundles}_{j_1}^{\text{Pes}}$ $\succ (L_{i_1}(n),\ldots,L_{i_1}(n))$

We first introduce some notation that will be useful to define the preference profile in Step 1 and Step 2. Let $\mathcal{O}(1) = (j_1, i_1)$. That is, agent j_1 is the first to choose an item in the sequential allocation, and she chooses from category D_{i_1}. Let L_{i_1} denote the order over $\{1,\ldots,n\}$ representing the order for the *agents* to choose items from D_{i_1} in $\mathcal{O}$. That is, $j \rhd_{L_{i_1}} j'$ if and only if $(j,i_1) \rhd_{\mathcal{O}} (j',i_1)$. By definition we have $j_1 = L_{i_1}(1)$. For any $j \leq n$, we let $Pred_{i_1}(j) = L_{i_1}(L_{i_1}^{-1}(j)-1)$ denote the predecessor of agent j in L_{i_1}, that is, the latest agent who chose an item from category i_1 before agent j chooses from category i_1. If $j=1$, then we let the last agent in L_{i_1} be her predecessor, that is, $Pred_{i_1}(1) = L_{i_1}(n)$.

Step 1: define bottom bundles. In order to match the upper bounds shown in the proof of Proposition 1, the bundles described in the proof of Proposition 1 must be the *only* bundles that are ranked below $(j,\ldots,j)$ by agent j. This is the part of the preference profile we will construct in the first step.

For all i and t, we first define $D^*_{i,t}$ to be the subset of $D_i = \{1,\ldots,n\}$ such that $q \in D^*_{i,t}$ if and only if agent q has not chosen an item from D_i before the t-th round. By definition, if $\mathcal{O}(t) = (j,i)$ then $j \in D^*_{i,t}$. Formally,

$$D^*_{i,t} = \{q \leq n : \mathcal{O}^{-1}(q,i) \geq t\}$$

We note that $D^*_{i,t}$ is defined solely by i, t, and $\mathcal{O}$, which means that it does not depend on agents' preferences and behavior in previous rounds. Later in this proof we will show that for our constructed preference profile, in each round (j,i) the active agent j will choose j from D_i, so that $D^*_{i,t}$ is the remaining items for category i at the beginning of round t of the sequential allocation.

For any $1 \leq l \leq p$, we let $t^*_{j,l} = \mathcal{O}^{-1}(j, \mathcal{O}_j(l))$. That is, $t^*_{j,l}$ is the round where agent j chooses an item from the l-th category in $\mathcal{O}_j$, which is not necessarily category l. For each agent j we specify their bottom bundles as follows.

— If agent j is optimistic, then we let the following bundles be ranked in the bottom of her preferences:

$$\text{BottomBundles}_j^{\text{Opt}} = (j_{\mathcal{O}_j(1)},\ldots,j_{\mathcal{O}_j(K_j-1)}) \times \prod_{l=K_j}^{p} D^*_{\mathcal{O}_j(l),t^*_{j,l}}, \quad (1)$$

where $(j,\ldots,j)$ is ranked on the top of these bundles, and the order over the remaining bundles is defined arbitrarily. It follows that $(j,\ldots,j)$ is ranked in the $(\prod_{l=K_j}^{p} k_{j,O_j(l)})$-th position from the bottom by agent j.

— If agent j is pessimistic, then we first define the following bundles:

$$\text{BottomBundles}_j^{\text{Pes}} = \bigcup_{l=1}^{p} \bigcup\nolimits_{d\in D^*_{\mathcal{O}_j(l),t^*_{j,l}}} \{([d]_{\mathcal{O}_j(l)},[j]_{-\mathcal{O}_j(l)})\}, \quad (2)$$

where $[j]_{-\mathcal{O}_j(l)}$ means that all components except the $\mathcal{O}_j(l)$-th component is j. Bundles in $\text{BottomBundles}_j^{\text{Pes}}$ are (partially) ranked as follows: first, $(j,\ldots,j)$ is ranked on the top; then, for any $1 \leq l_1 < l_2 \leq p$ and any $d_1 \in D^*_{\mathcal{O}_j(l_1),t^*_{j,l_1}}$ and $d_2 \in D^*_{\mathcal{O}_j(l_2),t^*_{j,l_2}}$ with $d_1 \neq j$ and $d_2 \neq j$, we rank $([d_1]_{\mathcal{O}_j(l_1)}, [j]_{-\mathcal{O}_j(l_1)})$ below $([d_2]_{\mathcal{O}_j(l_2)}, [j]_{-\mathcal{O}_j(l_2)})$.

— If $j \neq j_1$, then we simply let $\text{BottomBundles}_j^{\text{Pes}}$ (with the partial orders specified above) be the bundles ranked in the bottom position.

— If $j = j_1$, then we move $([Pred_{i_1}(j)]_{i_1}, [j]_{-i_1}) = ([L_{i_1}(n)]_{i_1}, [j_1]_{-i_1})$ to the bottom place and replace it by $(Pred_{i_1}(j), \ldots, Pred_{i_1}(j)) = (L_{i_1}(n), \ldots, L_{i_1}(n))$, and then let these be ranked in the bottom positions of agent j's preferences. That is, the bottom bundles are:

$$(j_1, \ldots, j_1) \succ (\text{BottomBundles}_j^{\text{Pes}} \setminus \{(j_1, \ldots, j_1), ([L_{i_1}(n)]_{i_1}, [j_1]_{-i_1})\}) \succ (L_{i_1}(n), \ldots, L_{i_1}(n))$$

In both cases $(j, \ldots, j)$ is ranked at the $(1 + \prod_{l=K_j}^{p}(k_{j,O_j(l)} - 1))$-th position from the bottom.

Step 2: define top bundles. We now specify the top two bundles (sometimes only the top bundle) for optimistic agents, and show that they are compatible with our constructions in Step 1. For any optimistic agent j:

— When $j \neq j_1$, there are following two cases:
  — case 1: $K_j = 1$. We let $([Pred_{i_1}(j)]_{i_1}, [j]_{-i_1})$ be the top-ranked bundle of agent j.
  — case 2: $K_j > 1$. We let $([Pred_{i_1}(j)]_{i_1}, [j]_{-i_1})$ be the top-ranked bundle of agent j. Moreover, if $i_1 \neq \mathcal{O}_j(K_j)$, then we rank $([Pred_{\mathcal{O}_j(K_j)}(j)]_{\mathcal{O}_j(K_j)}, [j]_{-\mathcal{O}_j(K_j)})$ at the second position. We recall that $Pred_{\mathcal{O}_j(K_j)}(j)$ is the predecessor of j in $L_{\mathcal{O}_j(K_j)}$, the order for the *agents* to choose items from $D_{\mathcal{O}_j(K_j)}$.

  These do not conflict the preferences specified in Step 1 because item $Pred_{i_1}(j)$ in D_{i_1} is not available for agent j when she is about to choose an item in D_{i_1}, and item $Pred_{\mathcal{O}_j(K_j)}(j)$ in $D_{\mathcal{O}_j(K_j)}$ is not available for agent j when she is about to choose an item in $D_{\mathcal{O}_j(K_j)}$. Hence, none of these bundles are in $\text{BottomBundles}_j^{\text{Opt}}$.

— When $j = j_1$, there are following two cases:
  — case 1: $K_j = 1$. Since $(j_1, i_1) = \mathcal{O}(1)$, for all i, $D^*_{i,\mathcal{O}^{-1}(j,i)} = D_i$, which means that agent j is guaranteed to get her top-ranked bundle after the sequential allocation. In this case we let $(j, \ldots, j)$ be agent j's top-ranked bundle and let $([L_{i_1}(n)]_{i_1}, [j]_{-i_1})$ be ranked in agent j's second position. These do not conflict the preferences specified in Step 1 because in this case Step 1 only specifies that $(j, \ldots, j)$ be ranked in the top position.
  — case 2: $K_j > 1$. We rank $([Pred_{\mathcal{O}_j(K_j)}(j)]_{\mathcal{O}_j(K_j)}, [j]_{-\mathcal{O}_j(K_j)})$ at the top position. We then rank $([L_{i_1}(n)]_{i_1}, [j]_{-i_1})$ at the second position. Since $i_1 = \mathcal{O}_j(1)$, we have $\mathcal{O}_j(K_j) \neq i_1$, otherwise $K_j = 1$. Hence, $([Pred_{\mathcal{O}_j(K_j)}(j)]_{\mathcal{O}_j(K_j)}, [j]_{-\mathcal{O}_j(K_j)}) \neq ([L_{i_1}(n)]_{i_1}, [j]_{-i_1})$. These do not conflict the preferences specified in Step 1 because category i_1 is agent j_1's first category in $\mathcal{O}_{j_1}$, which means that $i_1 < K_j$, thus $([L_{i_1}(n)]_{i_1}, [j]_{-i_1}) \notin \text{BottomBundles}_{j_1}^{\text{Opt}}$; also $Pred_{\mathcal{O}_j(K_j)}(j)$ is not available when agent j_1 is about to choose an item for category $\mathcal{O}_j(K_j)$, which means that $([Pred_{\mathcal{O}_j(K_j)}(j)]_{\mathcal{O}_j(K_j)}, [j]_{-\mathcal{O}_j(K_j)}) \notin \text{BottomBundles}_{j_1}^{\text{Opt}}$.

For any pessimistic agent j, we simply let her top-ranked bundle be $([Pred_{i_1}(j)]_{i_1}, [j]_{-i_1})$ (we recall that $Pred_{i_1}(j_1) = L_{i_1}(n)$). We claim that preferences specified in the second step do not conflict preferences specified in the first step for bottom bundles.

— If $j \neq j_1$, then we need to show that $([Pred_{i_1}(j)]_{i_1}, [j]_{-i_1}) \notin \text{BottomBundles}_j^{\text{Pes}}$. When agent j is about to choose her item from D_{i_1}, agent $Pred_{i_1}(j)$ has already chosen her item from D_{i_1}, which means that $Pred_{i_1}(j)$ is unavailable for agent j. This means that $([Pred_{i_1}(j)]_{i_1}, j_{-i_1}) \notin \text{BottomBundles}_j^{\text{Pes}}$.

— If $j = j_1$, then by definition (see Table III) $([L_{i_1}(n)]_{i_1}, [j_1]_{-i_1})$ is replaced by $(L_{i_1}(n), \ldots, L_{i_1}(n))$ in $\text{BottomBundles}_{j_1}^{\text{Pes}}$, which means that it can be ranked in the top.

Step 3: extend to full preference profile. For any j, let R_j be an arbitrary linear order over $\mathfrak{D}$ that satisfies all constraints defined in the previous two steps (see Table IV and V). Let $P = (R_1, \ldots, R_n)$.

We now show by induction on the round in the sequential allocation mechanism, denoted by t, that if we apply the sequential allocation $\mathcal{O}$ to P, then for all $j \leq n$, agent j gets $(j, \ldots, j)$.

When $t = 1$, agent j_1 chooses an item from D_{i_1}. If j_1 is optimistic, then it is not hard to check that the i_1-th component of the top-ranked bundle of R_{j_1} is j_1 (the top-ranked bundles are $(j, \ldots, j)$ and $([j']_{\mathcal{O}_j(K_j)}, [j]_{-\mathcal{O}_j(K_j)})$, for case 1 ($K_{j_1} = 1$) and case 2 ($K_{j_1} > 1$), respectively. If agent j_1 is pessimistic, then for any $d \in D_{i_1}$ with $d \neq j_1$, there exists a bundle whose i_1th component is d and is ranked below any bundle whose i_1th component is j_1. More precisely, if $d \neq Pred_{i_1}(j_1) = L_{i_1}(n)$, then such a bundle is $([d]_{i_1}, [j]_{-i_1})$; if $d = Pred_{i_1}(j_1) = L_{i_1}(n)$, then such an bundle is $(L_{i_1}(n), \ldots, L_{i_1}(n))$. In both cases a pessimistic agent j_1 will choose item j_1 from D_{i_1}.

Suppose in every round before round t, the active agent j chose item j from the designated category. Let $\mathcal{O}(t) = (j, i)$. If j is optimistic, then we show in the following four cases that she will choose item j from D_i in round t.

— $j \neq j_1$, $K_j = 1$. In this case j is guaranteed to get her top-ranked available bundle. It is not hard to check that the available bundles are a subset of $\text{BottomBundles}_j^{\text{Opt}}$, where $(j, \ldots, j)$ is available and is ranked in the top. Therefore agent j will choose item j.

— $j \neq j_1$, $K_j > 1$. There are following cases:

(1) Agent $Pred_{i_1}(j)$ has not chosen her item from D_{i_1}. In this case the top-ranked bundle $([Pred_{i_1}(j)]_{i_1}, [j]_{-i_1})$ is still available by the induction hypothesis.

(2) Agent $Pred_{i_1}(j)$ has chosen an item from D_{i_1} and $Pred_{\mathcal{O}_j(K_j)}(j)$ has not chosen her item from $D_{\mathcal{O}_j(K_j)}$. By the induction hypothesis, agent $Pred_{i_1}(j)$ chose item $Pred_{i_1}(j)$ from category D_{i_1}, which means that $([Pred_{i_1}(j)]_{i_1}, [j]_{-i_1})$ is unavailable. The bundle $([Pred_{\mathcal{O}_j(K_j)}(j)]_{\mathcal{O}_j(K_j)}, [j]_{-\mathcal{O}_j(K_j)})$ becomes the top-ranked available bundle due to the induction hypothesis, whose j-th component is j.

(3) $Pred_{i_1}(j)$ has chosen item $Pred_{i_1}(j)$ from D_{i_1} and $Pred_{\mathcal{O}_j(K_j)}(j)$ has chosen her item from $D_{\mathcal{O}_j(K_j)}$. In this case, we first claim that $\mathcal{O}_j^{-1}(i) \geq K_j$. For the sake of contradiction suppose $\mathcal{O}_j^{-1}(i) < K_j$. Then, by the definition of $Pred_{\mathcal{O}_j(K_j)}$, no agent chooses an item from $D_{\mathcal{O}_j(K_j)}$ between round $\mathcal{O}_j^{-1}(i)$ and t^*_{j,K_j}. We recall that t^*_{j,K_j} is the round when agent j chooses an item from $D_{\mathcal{O}_j(K_j)}$. However, this violates the minimality of K_j since no agent chooses an item from $D_{\mathcal{O}_j(K_j)}$ between round $t^*_{j,K_j-1} > \mathcal{O}_j^{-1}(i)$ and t^*_{j,K_j}. Hence, we must have that $\mathcal{O}_j^{-1}(i) \geq K_j$. By the induction hypothesis, the available bundles are a subset of $\text{BottomBundles}_j^{\text{Opt}}$ and $(j, \ldots, j)$ is still available and is ranked at the top, which means that agent j will choose item j from D_i.

In all three cases above, the ith component of the top-ranked available bundle is j, which means that agent j will choose item j.

— $j = j_1$, $K_j = 1$. By the induction hypothesis, the top-ranked bundle $(j, \dots, j)$ is still available, which means that agent j will choose item j.

— $j = j_1$, $K_j > 1$. If agent $Pred_{\mathcal{O}_j(K_j)}(j)$ has not chosen her item from $D_{\mathcal{O}_j(K_j)}$, then by the induction hypothesis the top bundle $([Pred_{\mathcal{O}_j(K_j)}(j)]_{\mathcal{O}_j(K_j)}, [j]_{-\mathcal{O}_j(K_j)})$ is still available and $i \neq \mathcal{O}_j(K_j)$. If agent $Pred_{\mathcal{O}_j(K_j)}(j)$ has chosen item $Pred_{\mathcal{O}_j(K_j)}(j)$ from $D_{\mathcal{O}_j(K_j)}$, then by the induction hypothesis the available bundles are a subset of $\text{BottomBundles}_j^{\text{Opt}}$ with $(j, \dots, j)$ ranked at the top. In both cases the ith component of the top-ranked available bundle is j. Therefore agent j will choose item j.

If agent j is pessimistic, then by the induction hypothesis the available items in D_i are $D^*_{i,t}$, and $j \in D^*_{i,t}$. For any $d \in D^*_{i,t}$ with $d \neq j$, $([d]_i, [j]_{-i})$ is still available and is ranked lower than any *available* bundle whose i-th component is j in $\text{BottomBundles}_j^{\text{Pes}}$. Therefore, a pessimistic agent j will choose item j in this round.

It follows that after the sequential allocation, for all $j \leq n$, agent j gets $(j, \dots, j)$. It is not hard to verify that condition 1 and 2 hold.

To show that condition 3 holds, consider the allocation where agent j gets $([Pred_{i_1}(j)]_{i_1}, [j]_{-i_1})$. In this allocation, all agents except j_1 get their top-ranked bundle, and j_1 gets her top-ranked bundle (if j_1 is pessimistic) or second-ranked bundle (if j_1 is optimistic). This proves the theorem. ■

Proposition 2. *Among all categorical sequential mechanisms, serial dictatorships with all-optimistic agents have the best (smallest) worst-case utilitarian rank and the worst (largest) worst-case egalitarian rank.*

Proof: The worst-case egalitarian rank of any serial dictatorship is n^p when all agents have the same preferences. To prove the optimality of worst-case utilitarian rank, given $f_{\mathcal{O}}$, we consider the multiset composed of the numbers of items in the designated category that the active agent can choose in each step. That is, we consider the multiset $\text{RI} = \{k_{j,l} : \forall j \leq n, l \leq p\}$. Since in each step in the execution of $f_{\mathcal{O}}$, only one item is allocated, RI is composed of p copies of $\{1, \dots, n\}$. Since for each agent j, $(1 + \sum_{l=1}^{p}(k_{j,O_j(l)} - 1)) \leq \prod_{l=1}^{p} k_{j,O_j(l)}$ (we note that in the right hand side, l starts with 1 but not K_j), the best worst-case utilitarian rank is at least $n(n^p + 1) - \sum_{j=1}^{n} \prod_{l=1}^{p} k_{j,O_j(l)} \geq n(n^p + 1) - \sum_{j=1}^{n} j^p$. It is not hard to verify that this lower bound is achieved by any serial dictatorship with all-optimistic agents. ■

Proposition 3. *Any categorical sequential mechanisms with all-optimistic agents has the worst (largest) worst-case egalitarian rank, which is n^p.*

Proof: By Theorem 2, the proposition is equivalent to the existence of an agent j such that for all $l \geq K_j$, $k_{j,\mathcal{O}_j(l)} = 1$. For the sake of contradiction, let us assume the following condition:

Condition (*): for every agent j, there exists $l \geq K_j$ such that $k_{j,\mathcal{O}_j(l)} > 1$.

Let $\mathcal{O}(np) = (j_n, i_p)$. It follows that $k_{j_n,i_p} = 1$ because there is only one item left. By condition (*), there exists i_{p-1} with $K_n \leq \mathcal{O}_{j_n}^{-1}(i_{p-1})$ such that $k_{j_n,i_{p-1}} > 1$. Let j_{n-1} denote the agent who is the last to chose an item from category i_{p-1}. We have $j_{n-1} \neq j_n$, because agent n is not the last agent to choose an item from category i_{p-1}. By definition, we have $k_{j_{n-1},i_{p-1}} = 1$. Moreover, $(j_{n-1}, \mathcal{O}_{j_{n-1}}(K_{j_{n-1}})) \rhd_{\mathcal{O}} (j_n, i_{p-1}) \rhd_{\mathcal{O}} (j_n, \mathcal{O}_{j_n}(K_{j_n}))$, which simply states that j_{n-1} chooses an item from category $\mathcal{O}_{j_{n-1}}(K_{j_{n-1}})$ after j_n chooses an item from category i_{p-1} (the second half of the inequality is due to the way we choose i_{p-1}). This inequality holds because if agent j_{n-1} chooses an item from category $\mathcal{O}_{j_{n-1}}(K_{j_{n-1}})$ before agent j_n chooses an item from category i_{p-1}, then agent j_n "interrupts" agent j_{n-1} from choosing an item from category i_{p-1}, which contradicts the definition of $K_{j_{n-1}}$.

By condition (*), there exists i_{p-2} such that $K_{j_{n-1}} \leq \mathcal{O}^{-1}_{j_{n-1}}(i_{p-2})$ and $k_{j_{n-1},i_{p-2}} > 1$. Similarly, we can define j_{n-2}, prove that $j_{n-2} \neq j_{n-1}$ and $(j_{n-2}, \mathcal{O}_{j_{n-2}}(K_{j_{n-2}})) \rhd_{\mathcal{O}} (j_{n-1}, i_{p-2}) \rhd_{\mathcal{O}} (j_{n-1}, \mathcal{O}_{j_{n-1}}(K_{j_{n-1}}))$.

However, this process cannot continue forever, since otherwise we will obtain an infinite sequence in $\mathcal{O}$: $(j_n, \mathcal{O}_{j_n}(K_{j_n})) \lhd_{\mathcal{O}} (j_{n-1}, \mathcal{O}_{j_{n-1}}(K_{j_{n-1}})) \lhd_{\mathcal{O}} (j_{n-2}, \mathcal{O}_{j_{n-2}}(K_{j_{n-2}})) \lhd_{\mathcal{O}} \cdots$, but np is finite. This leads to a contradiction. ■

Proposition 4. *For any even number p, the worst-case egalitarian rank of any balanced mechanism with all-pessimistic agents (see Example 2) is $n^p - (n-1)p/2$. These are the mechanisms with the best worst-case egalitarian rank among categorical sequential mechanisms with all-pessimistic agents.*

Proof: For any balanced mechanism, it is not hard to see that for any agent j, $\mathcal{O}_j = 1 \rhd \cdots \rhd p$. For any $l < p/2$ and any $j \leq n$, we have $k_{j,2l-1} + k_{j,2l} = n + 1$. Since all agents are pessimistic, by part 2 of Theorem 2, their worst-case ranks are all equal to $n^p - (n-1)p/2$. The optimality of balanced mechanisms come from the fact that for any categorical sequential mechanisms $\sum_{j,l} k_{j,l} = (n+1)np/2$. Therefore, there must exists an agent j^* with $\sum_{l=1}^{p} k_{j,l} \leq (n+1)p/2$. ■

Proposition 5. *For any even number p with $2^p > 1+(n-1)p/2$, there exists a categorical sequential mechanism with both optimistic and pessimistic agents, whose worst-case egalitarian rank is strictly better (smaller) than $n^p - (n-1)p/2$.*

Proof: We prove the proposition by explicitly constructing such a mechanism. The idea is, agents $\{1, \ldots, n-1\}$ choose the items as in a balanced categorial sequential mechanism for $n-1$ agents, then we let agent n "interrupt" them and choose all items in consecutive p rounds right before their last iteration, i.e. the last $(n-1)$ round. Then, we let agents 1 through $n-1$ be optimistic and let agent n be pessimistic. For example, when $n = 3$ and $p = 4$, the order is $(1,1) \rhd (2,1) \rhd (2,2) \rhd (1,2) \rhd (1,3) \rhd (2,3) \rhd (3,1) \rhd (3,2) \rhd (3,3) \rhd (3,4) \rhd (2,4) \rhd (1,4)$. Agent 1 and agent 2 are optimistic and agent 3 is pessimistic.

By part 2 of Theorem 2, for any agent $j \leq n-1$, the worst-case rank is $n^p + 1 - (1 + np/2)$. By part 1 of Theorem 2, the worst-case rank for agent n is $n^p + 1 - 2^p$. This proves the proposition. ■