Strategic Sequential Voting in Multi-Issue Domains and Multiple-Election Paradoxes
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Abstract

In many settings, a group of voters must come to a joint decision on multiple issues. In practice, this is often done by voting on the issues in sequence. We model sequential voting in multi-issue domains as a complete-information extensive-form game, in which the voters are perfectly rational and their preferences are common knowledge. In each step, the voters simultaneously vote on one issue, and the order of the issues is given exogenously before the process. We call this model *strategic sequential voting*.

We focus on domains characterized by multiple binary issues, so that strategic sequential voting leads to a unique outcome under a natural solution concept. We show that under some conditions on the preferences, this leads to the same outcome as truthful sequential voting, but in general it can result in very different outcomes. Moreover, we show that sometimes the order of the issues has a strong influence on the winner. Most significantly, we illustrate several *multiple-election paradoxes* in strategic sequential voting: there exists a profile for which the winner under strategic sequential voting is ranked nearly at the bottom in all votes, and the winner is Pareto-dominated by almost every other alternative. We show that changing the order of the issues cannot completely prevent such paradoxes. We also study the possibility of avoiding the paradoxes for strategic sequential voting by imposing some constraints on the profile, such as separability, lexicographicity or $\mathcal{O}$-legality. Finally, we investigate the existence of multiple election paradoxes for other common voting rules from a non-strategic perspective.

1 Introduction

In a traditional voting system, each voter is asked to report a linear order over the alternatives to represent her preferences. Then, a *voting rule* is applied to the resulting profile of reported preferences to select a winning alternative. In practice, the set of alternatives often has a *multi-issue* structure. That is, there are p issues $\mathcal{I} = \{\mathbf{x}_1, \dots, \mathbf{x}_p\}$, and each issue can take a value in a *local domain*. In other words, the set of alternatives is the Cartesian product of the local domains. For example, in *multiple referenda*, the inhabitants of a local district are asked to vote on multiple inter-related issues [5]. Another example is *voting by committees*, in which the voters select a subset of objects [1], where each object can be seen as a binary issue.

Voting in multi-issue domains has been extensively studied by economists, and more recently has attracted the attention of computer scientists. Previous work has focused on proposing a natural and compact *voting language* for the voters to represent their preferences, as well as designing a sensible voting rule to make decisions based on the reported preferences using such a language. A natural approach is to let voters vote on the issues separately, in the following way. For each issue (simultaneously, not sequentially), each voter reports her preferences for that issue, and then, a *local rule* is used to select the winning value that the issue will take. This voting process is called *issue-by-issue* or *seat-by-seat* voting.

Computing the winner for issue-by-issue voting rules is easy, and it only requires a modest amount of communication from the voters to the mechanism. Nevertheless, issue-by-issue voting has some drawbacks. First, a voter may feel uncomfortable expressing her preferences over one issue independently of the values that the other issues take [13]. It has been pointed out that issue-by-issue voting avoids this problem if the voters' preferences are *separable* (that is, for any issue i, regardless of the values for the other issues, the voter's preferences over issue i are always the same) [12]. Second, *multiple-election paradoxes* arise in issue-by-issue voting [5, 12, 19, 21]. In models that do not consider strategic (game-theoretic) voting, previous works have shown several types of paradoxes: sometimes the winner is a Condorcet loser; sometimes the winner is Pareto-dominated by another alternative (that is, that alternative is preferred to the winner in all votes); and sometimes the winner is ranked in a very low position by all voters.

One way to partly escape these paradoxes consists in organizing the multiple elections *sequentially*: given an order $\mathcal{O}$ over all issues (without loss of generality, we take $\mathcal{O}$ to be $\mathbf{x}_1 > \cdots > \mathbf{x}_n$), the voters first vote on issue $\mathbf{x}_1$; then, the value collectively chosen for $\mathbf{x}_1$ is determined using some voting rule and broadcast to the voters, who then vote on issue $\mathbf{x}_2$, and so on. When the issues are all binary, it is natural to choose the majority rule at each stage (plus, in the case of an even number of voters, some tie-breaking mechanism). Such processes are conducted in many real-life situations. For instance, suppose there is a full professor position and an assistant professor position to be filled. Then, it is realistic to expect that the committee will first decide who gets the full professor position. Another example is that at the executive meeting of the co-owners of a building, important decisions like whether a lift should be installed or not, how much money should be spent to repair the roof are usually taken before minor decisions. In each of these cases, it is clear that the decision made on one issue influences the votes on later issues, thus the order in which the issues are decided potentially has a strong influence on the final outcome.

Now, if voters are assumed to know the preferences of other voters well enough, then we can expect them to vote strategically at each step, forecasting the outcome at later steps conditional on the outcomes at earlier steps. Let us consider the following example.

Example 1 *Three residents want to vote to decide whether they should build a swimming pool and/or a tennis court. There are two issue S and T. S can take the value of s (meaning "to build the swimming pool") or $\bar{s}$ (meaning "not to build the swimming pool"). Similarly, T takes a value in $\{t, \bar{t}\}$. Suppose the preferences of the three voters are, respectively, $st \succ \bar{s}t \succ s\bar{t} \succ \bar{s}\bar{t}$, $s\bar{t} \succ st \succ \bar{s}t \succ \bar{s}\bar{t}$ and $\bar{s}t \succ \bar{s}\bar{t} \succ s\bar{t} \succ st$. Voter 2 and 3 took the budget constraint into consideration so that they do not rank st as their first choices. Suppose the voters first vote on issue S then on T. Moreover, since both issues are binary, the local rule used at each step is majority (there will be no ties, because the number of voters is odd). Voter 1 is likely to reason in the following way:* if the outcome of the first step is s, then voters 2 and 3 will vote for $\bar{t}$, since they both prefer $s\bar{t}$ to st, and the final outcome will be $s\bar{t}$; now, if the outcome of the first step is $\bar{s}$, then voters 2 and 3 will vote for t, and the final outcome will be $\bar{s}t$; because I prefer $\bar{s}t$ to $s\bar{t}$, I am better off voting for $\bar{s}$, since either it will not make any difference, or it will; and in the latter case it will lead to a final outcome of $\bar{s}t$ instead of $s\bar{t}$. *If voters 2 and 3 reason in the same way, then 2 will vote for s and 3 for $\bar{s}$; hence, the result of the first step is $\bar{s}$, and then, since two voters out of three prefer $\bar{s}t$ to $s\bar{t}$, the final outcome will be $\bar{s}t$. Note that the result is fully determined, provided that (1) it is common knowledge that voters behave strategically according to the principle we have stated informally, (2) the order in which the issues are decided, as well as the local voting rules used in all steps, are also common knowledge, and (3) voters' preferences are common knowledge. Therefore, these three assumptions allow the voters and the modeler (provided he knows as much as the voters) to predict the final outcome.*

Let us take a closer look at voter 1 in Example 1. Her preferences are *separable*: she prefers s to $\bar{s}$ whatever the value of T, and t to $\bar{t}$ whatever the value of S. *And yet she strategically votes for* $\bar{s}$, because the outcome for S affects the outcome for T. Moreover, while voters 2 and 3 have nonseparable preferences, still, all three voters' preferences enjoy the following property: their preferences over the value of S are independent of the value of T. Such a profile is called a *legal* profile with respect to the order $S > T$, meaning that the voters vote on S first, then on T. Lang and Xia [13] defined a family of sequential voting rules on multi-issue domains, restricted to $\mathcal{O}$-legal profiles for some order $\mathcal{O}$ over the issues, where at each step, each voter is expected to vote for her preferred value for the issue $\mathbf{x}_i$ under consideration given the values of all issues decided so far [1]; then, the value of $\mathbf{x}_i$ is chosen according to a local voting rule, and this local outcome is broadcast to the voters. Note that for the profile given in Example 1, suppose the local rule used to decide an issue is always majority, the outcome of the first step will be s (since two voters out of three prefer s to $\bar{s}$, unconditionally), and the final outcome will be $s\bar{t}$, which is different from the outcome we obtain if voters behave strategically. The reason for this discrepancy is that in [13], voters are not assumed to know the others' preferences and are assumed to vote truthfully.

We have seen that even if the voters' preferences *are* $\mathcal{O}$-legal, voters may in fact

[1]The $\mathcal{O}$-legality condition ensures that this notion of 'preferred value of $\mathbf{x}_i$' is meaningful.

have no incentive to vote truthfully. Consequently, existing results on multiple-election paradoxes are not directly applicable to situations where voters vote strategically.

Our contributions

In this paper, we analyze the complete-information game-theoretic model of sequential voting that we illustrated in Example 1. This model applies to any preferences that the voters may have (not just $\mathcal{O}$-legal ones), though they must be strict orders on the set of all alternatives.

We focus on voting in multi-binary-issue domains, that is, for any $i \leq p$, $\mathbf{x}_i$ must take a value in $\{0_i, 1_i\}$. This has the advantage that for each issue, we can use the majority rule as the local rule for that issue. We use a game-theoretic model to analyze outcomes that result from sequential voting. Specifically, we model the sequential voting process as a p-stage complete-information game, as follows. There is an order $\mathcal{O}$ over all issues (without loss of generality, let $\mathcal{O} = \mathbf{x}_1 > \mathbf{x}_2 > \cdots > \mathbf{x}_p$), which indicates the order in which these issues will be voted on. For any $1 \leq i \leq p$, in stage i, the voters vote on issue $\mathbf{x}_i$ simultaneously, and the majority rule is used to choose the winning value for $\mathbf{x}_i$. We make the following game-theoretic assumptions: it is common knowledge that all voters are perfectly rational; the order $\mathcal{O}$ and the fact that in each step, the majority rule is used to determine the winner are common knowledge; all voters' preferences are common knowledge.

We can solve this game by a type of backward induction already illustrated in Example 1: in the last (pth) stage, only two alternatives remain (corresponding to the two possible settings of the last issue), so at this point it is a weakly dominant strategy for each voter to vote for her more preferred alternative of the two. Then, in the second-to-last ($(p-1)$th) stage, there are two possible local outcomes for the $(p-1)$th issue; for each of them, the voters can predict which alternative will finally be chosen, because they can predict what will happen in the pth stage. Thus, the $(p-1)$th stage is effectively a majority election between two alternatives, and each voter will vote for her more preferred alternative; etc. We call such a procedure the *strategic sequential voting procedure (SSP)*.

Given exogenously the order $\mathcal{O}$ over the issues, this game-theoretic analysis maps every profile of strict ordinal preferences to a unique outcome. Since any function from profiles of preferences to alternatives can be interpreted as a voting rule, the voting rule that corresponds to SSP is denoted by $SSP_{\mathcal{O}}$.

After the introduction of SSP, we show that, unfortunately, multiple-election paradoxes also arise under SSP. To better present our results, we introduce a parameter which we call the *minimax satisfaction index (MSI)*. For an election with m alternatives and n voters, it is defined in the following way. For each profile, consider the highest position that the winner obtains across all input rankings of the alternatives (corresponding to the most-satisfied voter); this is the *maximum satisfaction index* for this profile. Then, the minimax satisfaction index is obtained by taking the minimum over all profiles of the maximum satisfaction index. A low minimax satisfaction index means that there exists a profile in which the winner is ranked in low positions in all votes, thus indicating a multiple-election paradox. Our main theorem is the following.

Theorem 2 *For any $p \in \mathbb{N}$ and any $n \geq 2p^2 + 1$, the minimax satisfaction index of SSP when there are $m = 2^p$ alternatives and n voters is $\lfloor p/2 + 2 \rfloor$. Moreover, in the profile P that we use to prove the upper bound, the winner $SSP_{\mathcal{O}}(P)$ is Pareto-dominated by $2^p - (p + 1)p/2$ alternatives.*

We note that an alternative c Pareto-dominates another alternative c' implies that c beats c' in their pairwise election. Therefore, Theorem 2 implies that the winner for SSP is an almost Condorcet loser. It follows from this theorem that SSP exhibits all three types of multiple-election paradoxes: the winner is ranked almost in the bottom in every vote, the winner is an almost Condorcet loser, and the winner is Pareto-dominated by almost every other alternative. We further show a paradox (Theorem 3) that states that there exists a profile such that for *any* order $\mathcal{O}$ over the issues, for every voter, the SSP winner w.r.t. $\mathcal{O}$ is ranked almost in the bottom position. We also show that even when the voters' preferences can be represented by CP-nets that are compatible with a common order, multiple-election paradoxes still arise.

To see if there are similar paradoxes for common voting rules, where voters are assumed to vote truthfully, we calculate the minimax satisfaction index for some common voting rules, including dictatorships, positional scoring rules (including k-approval and Borda), plurality with runoff, Copeland, maximin, STV, Bucklin, ranked pairs, and (not necessarily balanced) voting trees. We show that for k-approval with a large k, voting trees, Copeland$_0$, and maximin we can find a similar paradox, and for the others there are no such paradoxes.

Related work and discussion

Our setting is closely related to the *multi-stage sophisticated voting*, studied by McKelvey and Niemi [15], Moulin [16], and Gretlein [11]. They investigated the model where the backward induction outcomes correspond to the truthful outcomes of voting trees. Therefore, our SSP is a special case of multi-stage sophisticated voting. However, their work focused on the characterization of the outcomes as the outcomes in the *sophisticated voting* [8], therefore did not shed much light on the quality of the equilibrium outcome. We, on the other hand, are primarily interested in the strategic outcome of the natural procedure of voting sequentially over multiple issues. Also, the relationship between sequential voting and voting trees takes a particularly natural form in the context of domains with multiple binary issues, as we will show. More importantly, we illustrate several multiple-election paradoxes for SSP, indicating that the equilibrium outcome could be extremely undesirable.

Another paper that is closely related to part of this work was written by Dutta and Sen [7]. They showed that social choice rules corresponding to binary voting trees can be implemented via backward induction via a sequential voting mechanism. This is closely related to the relationship revealed for multi-stage sophisticated voting and will also be mentioned later in this paper, that is, an equivalence between the outcome of strategic behavior in sequential voting over multiple binary issues, and a particular type of voting tree. It should be pointed out that the sequential mechanism that Dutta and Sen consider is somewhat different from sequential voting as we consider it—in particular, in the Dutta-Sen mechanism, one voter moves at a time, and a move consists not of a vote, but rather of choosing the next player to move (or in some states, choosing the winner).

Nevertheless, the approach by Dutta and Sen and our approach are related at a high level, though they are motivated quite differently: Dutta and Sen are interested in social choice rules corresponding to voting trees, and are trying to create sequential mechanisms that implement them via backward induction. We, on the other hand again, are primarily interested in the strategic outcome of the natural mechanism for voting sequentially over multiple issues, and use voting trees merely as a useful tool for analyzing the outcome of this process.

Less closely related, implementation by voting trees has previously been studied: Fischer et al. [9] consider the known result that the Copeland rule (which we define later in this paper) cannot be implemented by a voting tree [18], and set out to *approximate* the Copeland score using voting trees.

It has been pointed out that typical multiple-election paradoxes partly come from the incompleteness of information about the preferences of the voters [12]. However, the paradoxes in this paper show that assuming that voters' preferences are common knowledge does not allow to get rid of multiple election paradoxes. Another interpretation of these results is that we may need to move beyond sequential voting to properly address voting in multi-issue domains. However, note that other approaches than sequential voting may be extremely costly in terms of communication and computation, which comes down to saying, one more time, that voting on multiple related issues is an extremely challenging problem for which probably no perfect solution exists.

2 Preliminaries

2.1 Basics of voting

Let $\mathcal{X}$ be the set of *alternatives*, $|\mathcal{X}| = m$. A vote is a linear order over $\mathcal{X}$. The set of all linear orders over $\mathcal{X}$ is denoted by $L(\mathcal{X})$. For any $c \in \mathcal{X}$ and $V \in L(\mathcal{X})$, we let $rank_V(c)$ denote the position of c in V from the top. An *n-profile* P is a collection of n votes for some $n \in \mathbb{N}$, that is, $P \in L(\mathcal{X})^n$. For any $c, d \in \mathcal{X}$ and any profile P, we say c *Pareto-dominates* d, if for any $V \in P$, c is ranked higher than d in V, that is, $c \succ_V d$. A *voting rule* r is a mapping that assigns to each profile a unique winning alternative. That is, $r : L(\mathcal{X}) \cup L(\mathcal{X})^2 \cup \cdots \to \mathcal{X}$. Some common voting rules are listed below.

- *Dictatorships:* for every $n \in \mathbb{N}$ there exists a voter $j \leq n$ such that the winner is always the alternative that is ranked in the top position in V_j.

- *(Positional) scoring rules*: Given a *scoring vector* $\vec{v} = (v(1), \ldots, v(m))$, for any vote $V \in L(\mathcal{X})$ and any $c \in \mathcal{X}$, let $s(V, c) = v(i)$, where i is the rank of c in V. For any profile $P = (V_1, \ldots, V_n)$, let $s(P, c) = \sum_{j=1}^{n} s(V_j, c)$. The rule will select $c \in \mathcal{X}$ so that $s(P, c)$ is maximized. Some examples of positional scoring rules are *Borda*, for which the scoring vector is $(m-1, m-2, \ldots, 0)$, k-approval (App_k, with $k \leq m$), for which the scoring vector is $(\underbrace{1, \ldots, 1}_{k}, 0, \ldots, 0)$, *plurality*, for which the scoring vector is $(1, 0, \ldots, 0)$, and *veto*, for which the scoring vector is $(1, \ldots, 1, 0)$.

- *Copeland*$_\alpha$ ($0 \leq \alpha \leq 1$): For any two alternatives c and d, we can simulate a *pairwise election* between them, by seeing how many votes prefer c to d, and how many prefer d to c; the winner of the pairwise election is the one preferred more often. Then, an alternative receives one point for each win in a pairwise election, α points for each draw, and 0 point for each loss. The winner is the alternative that has the highest total score.

- *Plurality with runoff* ($Pluo$): The election has two rounds. In the first round, the alternatives are ranked from high to low according to the number of times they are ranked in the top position in the votes of the profile (that is, according to their plurality scores). Only the top two alternatives enter the second (runoff) round. In the runoff, we simulate a pairwise election between these two alternatives, and the alternative that wins the pairwise election is the winner.

- *Maximin*: Let $N(c, d)$ denote the number of votes that rank c ahead of d. The winner is the alternative c that maximizes $min\{N(c, c') : c' \in \mathcal{X}, c' \neq c\}$.

- *STV*: The election has $m - 1$ rounds. In each round, we count for each remaining alternative how many votes rank it highest among the remaining alternatives; then, the alternative with the lowest count drops out. The last remaining alternative is the winner.

- *Bucklin*: An alternative c's Bucklin score is the smallest number k such that more than half of the voters rank c among their top k alternatives. The winner is the alternative that has the lowest Bucklin score. If multiple alternatives have the lowest score k, then ties are broken by the number of voters who rank an alternative among their top k alternatives.

- *Ranked pairs*: This rule first creates an entire ranking of all the alternatives. $N(c, d)$ is defined as for the maximin rule. In each step, we will consider a pair of alternatives c, d that we have not previously considered; specifically, we choose the remaining pair with the highest $N(c, d)$. We then fix the ordering $c \succ d$, unless it contradicts orderings that we fixed previously (that is, it violates transitivity). We continue until all pairs of alternatives are considered (hence we end up with a full ranking). The alternative at the top of the ranking wins.

- *Voting trees*: A voting tree is a binary tree with m leaves, where each leaf is associated with an alternative. In each round, there is a pairwise election between an alternative c and its sibling d: if the majority of voters prefer c to d, then d is eliminated, and c is associated with the parent of these two nodes; similarly, if the majority of voters prefer d to c, then c is eliminated, and d is associated with the parent of these two nodes. The alternative that is associated with the root of the tree (wins all its rounds) is the winner.

2.2 Multi-issue domains

In this paper (except Section 7), the set of all alternatives $\mathcal{X}$ is a *multi-binary-issue domain*. That is, let $\mathcal{I} = \{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ ($p \geq 2$) be a set of *issues*, where each issue $\mathbf{x}_i$ takes a value in a binary *local domain* $D_i = \{0_i, 1_i\}$. The set of alternatives is $\mathcal{X} = D_1 \times \cdots \times D_p$, that is, an alternative is uniquely identified by its values on all issues. For any $Y \subseteq \mathcal{I}$ we denote $D_Y = \prod_{\mathbf{x}_i \in Y} D_i$.

Given a preference relation $\succ$ in $L(\mathcal{X})$, an issue $\mathbf{x}_i$, and a subset of issues $W \subseteq \mathcal{I}$, let $U = \mathcal{I} \setminus (W \cup \{\mathbf{x}_i\})$; then, $\mathbf{x}_i$ is *preferentially independent of W given U (with respect to $\succ$)* if for any $\vec{u} \in D_U$, any $a_i, b_i \in D_i$, and any $\vec{w}, \vec{w}' \in D_W$, $(\vec{u}, a_i, \vec{w}) \succ (\vec{u}, b_i, \vec{w})$ if and only if $(\vec{u}, a_i, \vec{w}') \succ (\vec{u}, b_i, \vec{w}')$. Informally, if we wish to find out whether changing the value of $\mathbf{x}_i$ from a_i to b_i (while keeping everything else fixed) will make the voter better or worse off, we only need to know the values of the issues in U.

Let $\mathcal{O} = \mathbf{x}_1 > \cdots > \mathbf{x}_p$. A preference relation $\succ$ is *$\mathcal{O}$-legal* if for any $i \leq p$, $\mathbf{x}_i$ is preferentially independent of $\{\mathbf{x}_{i+1}, \ldots, \mathbf{x}_p\}$ given $\{\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}\}$. In words, to find out whether a particular change in the value of an issue will make the voter better or worse off, we only need to know the values of earlier issues. A preference relation $\succ$ is *separable* if for any $i \leq p$, $\mathbf{x}_i$ is preferentially independent of $\mathcal{X} \setminus \{\mathbf{x}_i\}$. That is, to find out whether a particular change in the value of an issue will make the voter better or worse off, we do not need to know the value of any other issue. It follows directly that a separable preference relation is $\mathcal{O}$-legal for any $\mathcal{O}$.

A preference relation $\succ$ is *$\mathcal{O}$-lexicographic* if for any $i \leq p$, any $\vec{u} \in D_1 \times \cdots \times D_{i-1}$, any $a_i, b_i \in D_i$, and any $\vec{d}_1, \vec{d}_2, \vec{e}_1, \vec{e}_2 \in D_{i+1} \times \cdots \times D_p$, $(\vec{u}, a_i, \vec{d}_1) \succ (\vec{u}, b_i, \vec{e}_1)$ if and only if she prefers $(\vec{u}, a_i, \vec{d}_2) \succ (\vec{u}, b_i, \vec{e}_2)$. In words, if a profile is $\mathcal{O}$-lexicographic, then it is $\mathcal{O}$-legal, and moreover, earlier issues are more important—that is, to compare two alternatives, it suffices to know the values of the issues up to and including the first issue $\mathbf{x}_i$ on which they differ. (While the values of $\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}$ will be the same, they still matter in that they affect the voter's preferences on $\mathbf{x}_i$.) We note that $\mathcal{O}$-lexicographicity and separability are incomparable notions. For example, $0_1 0_2 \succ 1_1 0_2 \succ 0_1 1_2 \succ 1_1 1_2$ is separable (flipping 1_1 or 1_2 always makes the alternative rank higher) but not ($\mathbf{x}_1 > \mathbf{x}_2$)-lexicographic ($0_1 0_2 \succ 1_1 1_2$ but $1_1 0_2 \succ 0_1 1_2$). On the other hand, $0_1 0_2 \succ 0_1 1_2 \succ 1_1 1_2 \succ 1_1 0_2$ is ($\mathbf{x}_1 > \mathbf{x}_2$)-lexicographic but not separable.

A profile is separable/$\mathcal{O}$-lexicographic/$\mathcal{O}$-legal if it is composed of preference relations that are all separable/$\mathcal{O}$-lexicographic/$\mathcal{O}$-legal.

We can now define sequential composition of local voting rules. Given a vector of *local rules* $(r_1, \ldots, r_p)$ (where for any $i \leq p$, r_i is a voting rule for preferences over D_i), the *sequential composition* of $r_1, \ldots, r_p$ with respect to $\mathcal{O}$, denoted by $Seq_{\mathcal{O}}(r_1, \ldots, r_p)$, is defined for all $\mathcal{O}$-legal profiles as follows: $Seq_{\mathcal{O}}(r_1, \ldots, r_p)(P) = (d_1, \ldots, d_p) \in \mathcal{X}$, where for any $i \leq p$, $d_i = r_i(P|_{\mathbf{x}_i : d_1 \cdots d_{i-1}})$, where $P|_{\mathbf{x}_i : d_1 \cdots d_{i-1}}$ is composed of the voters' local preferences over $\mathbf{x}_i$, given that the issues preceding it take values $d_1, \ldots, d_{i-1}$. Thus, the winner is selected in p steps, one for each issue, in the following way: in step i, d_i is selected by applying the local rule r_i to the preferences of voters over D_i, conditioned on the values $d_1, \ldots, d_{i-1}$ that have already been determined for the issues that precede $\mathbf{x}_i$. In this paper, we only consider the case where every r_i is the majority rule over two alternatives, because when there are only two alternatives, the majority rule coincides with all voting rules mentioned above (except dictatorships).

3 Strategic sequential voting

3.1 Formal definition

Sequential voting on multi-issue domains can be seen as a game where in each step, the voters decide whether to vote for or against the issue under consideration after reasoning about what will happen next. We make the following assumptions.

1. All voters act strategically (in an optimal manner that will be explained later), and this is common knowledge.
2. The order in which the issues will be voted upon, as well as the local voting rules used at the different steps (namely, majority rules), are common knowledge.
3. All voters' preferences on the set of alternatives are common knowledge.

Assumption 1 is standard in game theory. Assumption 2 merely means that the rule has been announced. Assumption 3 (complete information) is the most significant assumption. It may be interesting to consider more general settings with incomplete information, resulting in a Bayesian game. Nevertheless, because the complete-information setting is a special case of the incomplete-information setting (where the prior distribution is degenerate), *all negative results obtained for the complete-information setting also apply to the incomplete-information setting*. That is, the restriction to complete information only strengthens negative results.

Given these assumptions, the voting process can be modeled as a game that is composed of p stages where in each stage, the voters vote simultaneously on one issue. Let $\mathcal{O}$ be the order over the set of issues, which without loss of generality we assume to be $\mathbf{x}_1 > \cdots > \mathbf{x}_p$. Let P be the profile of preferences over $\mathcal{X}$. The game is defined as follows: for each $i \leq p$, in stage i the voters vote simultaneously on issue i; then, the value of $\mathbf{x}_i$ is determined by the majority rule (plus, in the case of an even number of voters, some tie-breaking mechanism), and this local outcome is broadcast to all voters.

We now show how to solve the game. Because of assumptions 1 to 3, at step i the voters vote strategically, by recursively figuring out what the final outcome will be if the local outcome for $\mathbf{x}_i$ is 0_i, and what it will be if it is 1_i. More concretely, suppose that steps 1 to $i-1$ resulted in issues $\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}$ taking the values $d_1, \ldots, d_{i-1}$, and let $\vec{d} = (d_1, \ldots, d_{i-1})$. Suppose also that if $\mathbf{x}_i$ takes the value 0_i (respectively, 1_i), then, recursively, the remaining issues will take the tuple of values $\vec{a}$ (respectively, $\vec{b}$). Then, $\mathbf{x}_i$ is determined by a pairwise comparison between $(\vec{d}, 0_i, \vec{a})$ and $(\vec{d}, 1_i, \vec{b})$ in the following way: if the majority of voters prefer $(\vec{d}, 0_i, \vec{a})$ over $(\vec{d}, 1_i, \vec{b})$, then $\mathbf{x}_i$ takes the value 0_i; in the opposite case, $\mathbf{x}_i$ takes the value 1_i. This process, which corresponds to the strategic behavior in the sequential election, is what we call the *strategic sequential voting (SSP)* procedure, and for any profile P, the winner with respect to the order $\mathcal{O}$ is denoted by $SSP_{\mathcal{O}}(P)$.

As we shall see later, SSP can not only be thought of as the strategic outcome of sequential voting, but also as a voting rule in its own right. The following definition and two propositions merely serve to make the game-theoretic solution concept that we use precise; a reader who is not interested in this may safely skip them.

Definition 1 *Consider a finite extensive-form game which transitions among* states*. In each nonterminal state s, all players simultaneously take an action; this joint local action profile $(a_1^s, \ldots, a_n^s)$ determines the next state s'.*[2] *Terminal states t are associated with payoffs for the players (alternatively, players have ordinal preferences over the terminal states). The current state is always common knowledge among the players.*[3]

Suppose that in every final nonterminal state s (that is, every state that has only terminal states as successors), every player i has a (weakly) dominant action a_i^s. At each final nonterminal state, its local profile of dominant actions $(a_1^s, \ldots, a_n^s)$ results in a terminal state $t(s)$ and associated payoffs. We then replace each final nonterminal state s with the terminal state $t(s)$ that its dominant-strategy profile leads to. Furthermore suppose that in the resulting smaller tree, again, in every final nonterminal state, every player has a (weakly) dominant strategy. Then, we can repeat this procedure, etc. *If we can repeat this all the way to the root of the tree, then we say that the game is solvable by* within-state dominant-strategy backward induction (WSDSBI).

We note that the backward induction in perfect-information extensive-form games is just the special case of WSDSBI where in each state only one player acts.

Proposition 1 *If a game is solvable by WSDSBI, then the solution is unique.*

Proposition 2 *The complete-information sequential voting game with binary issues (with majority as the local rule everywhere) is solvable by WSDSBI when voters have strict preferences over the alternatives.*

Proposition 1 is obviously true. All proofs that are not in the main text can be found in the appendix.

We note that SSP corresponds to a particular balanced voting tree, as illustrated in Figure 1 for the case $p = 3$. In this voting tree, in the first round, each alternative is paired up against the alternative that differs only on the pth issue; each alternative that wins the first round is then paired up with the unique other remaining alternative that differs only on the $(p-1)$th and possibly the pth issue; etc. This bottom-up procedure corresponds exactly to the backward induction (WSDSBI) process.

Of course, there are many voting trees that do *not* correspond to an SSP election; this is easily seen by observing that there are only $p!$ different SSP elections (corresponding to the different orders of the issues), but many more voting trees. The voting tree corresponding to the order $\mathcal{O} = \mathbf{x}_1 > \cdots > \mathbf{x}_p$ is defined by the property that for any node v whose depth is i (where the root has depth 1), the alternative associated with any leaf in the left (respectively, right) subtree of v gives the value 0_i (respectively, 1_i) to $\mathbf{x}_i$.

[2] In the extensive-form representation of the game, each state is associated with multiple nodes, because in the extensive form only one player can move at a node.

[3] Hence, the only imperfect information in the extensive form of the game is due to simultaneous moves within states.

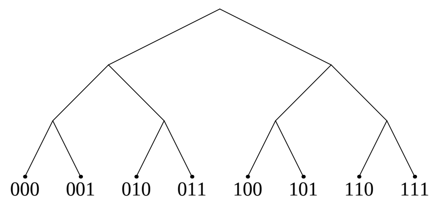

Figure 1: A voting tree that is equivalent to the strategic sequential voting procedure ($p = 3$). 000 is the abbreviation for $0_1 0_2 0_3$, etc.

3.2 Strategic sequential voting vs. truthful sequential voting

We have seen on Example 1 that even when the profile P is $\mathcal{O}$-legal, $SSP_{\mathcal{O}}(P)$ can be different from $Seq_{\mathcal{O}}(maj, \ldots, maj)(P)$. This means that even if the profile is $\mathcal{O}$-legal, voters may be better off voting strategically than truthfully. However, $SSP_{\mathcal{O}}(P)$ and $Seq_{\mathcal{O}}(maj, \ldots, maj)(P)$ are guaranteed to coincide under the further restriction that P is $\mathcal{O}$-lexicographic.

Proposition 3 *For any $\mathcal{O}$-lexicographic profile P, $SSP_{\mathcal{O}}(P) = Seq_{\mathcal{O}}(maj, \ldots, maj)(P)$.*

The intuition for Proposition 3 is as follows: if P is $\mathcal{O}$-lexicographic, then, as is shown in the proof of the proposition, when voters vote strategically under sequential voting (the Seq process), they are best off voting according to their true preferences in each round (their preferences in each round are well-defined because voters have $\mathcal{O}$-legal preferences in this case). When voters with $\mathcal{O}$-legal preferences vote truthfully in each round under sequential voting, the outcome is $Seq_{\mathcal{O}}(maj, \ldots, maj)(P)$; when they vote strategically, the outcome is $SSP_{\mathcal{O}}(P)$; and so, these must be the same when preferences are $\mathcal{O}$-lexicographic.

Now, there is another interesting domain restriction under which $SSP_{\mathcal{O}}(P)$ and $Seq(maj, \ldots, maj)(P)$ coincide, namely when P is $inv(\mathcal{O})$- legal, where $inv(\mathcal{O}) = (\mathbf{x}_p > \cdots > \mathbf{x}_1)$.

Proposition 4 *Let $inv(\mathcal{O}) = \mathbf{x}_p > \cdots > \mathbf{x}_1$. For any $inv(\mathcal{O})$-legal profile P, $SSP_{\mathcal{O}}(P) = Seq_{inv(\mathcal{O})}(maj, \ldots, maj)(P)$.*

As a consequence, when P is separable, it is *a fortiori* $inv(\mathcal{O})$-legal, and therefore, $SSP_{\mathcal{O}}(P) = Seq_{inv(\mathcal{O})}(maj, \ldots, maj)(P)$, which in turn is equal to $Seq_{\mathcal{O}}(maj, \ldots, maj)(P)$ and coincides with seat-by-seat voting [2].

Corollary 1
If P is separable, then $SSP_{\mathcal{O}}(P) = Seq_{\mathcal{O}}(maj, \ldots, maj)(P)$.

3.3 The winner is sensitive to the order over the issues

In the definition of SSP we simply fixed the order $\mathcal{O}$ to be $\mathbf{x}_1 > \mathbf{x}_2 > \cdots > \mathbf{x}_p$. Therefore, one natural question to ask is, for a given profile, to what extend it change the SSP winner if we adopt an order $\mathcal{O}'$ over the issues that is different from $\mathcal{O}$?

A first observation is that, as we mentioned briefly in the end of Section 3.1, there are $p!$ different SSP elections, each of which corresponds to an order over p issues. Therefore, a trivial upper bound on the number of alternatives that can be made to win in SSP by adopting a different order is $p!$. We recall that there are 2^p alternatives, which means that the $p!$ upper bound is only interesting when $p! < 2^p$, that is, $p \leq 3$. Example 2 shows that when $p = 2$ or $p = 3$, this trivial upper bound is actually tight. That is, there exists a profile such that by changing the order over the issues, $p!$ different alternatives can be made to win. Due to McGarvey's theorem [14], given any directed complete simple graph G over the alternatives, there exists a profile such that there is an edge from c to c' in G if and only if c beats c' in their pairwise elections. Such a graph is called the *majority graph* of the profile. Hence, in the example, we only show the majority graph, instead of explicitly constructing the full profile.

Example 2 *The majority graphs of the profiles for $p = 2$ and $p = 3$ are shown in Figure 2. Let P (respectively, P') denote an arbitrary profile whose majority graph is the same as Figure 2(a) (respectively, Figure 2(b)). It is not hard to verify that $SSP_{\mathbf{x}_1>\mathbf{x}_2}(P) = 00$ and $SSP_{\mathbf{x}_2>\mathbf{x}_1}(P) = 01$. For P', $SSP_{\mathcal{O}}(P')$ is summarized in Table 1. Note that $2! = 2$ and $3! = 6$. It follows that when $p = 2$ or $p = 3$, there exists a profile for which the SSP winners w.r.t. different orders over the issues are all different from each other.*

The order	$\mathbf{x}_1 > \mathbf{x}_2 > \mathbf{x}_3$	$\mathbf{x}_1 > \mathbf{x}_3 > \mathbf{x}_2$	$\mathbf{x}_2 > \mathbf{x}_1 > \mathbf{x}_3$
SSP winner	101	100	110
The order	$\mathbf{x}_2 > \mathbf{x}_3 > \mathbf{x}_1$	$\mathbf{x}_3 > \mathbf{x}_1 > \mathbf{x}_2$	$\mathbf{x}_3 > \mathbf{x}_2 > \mathbf{x}_1$
SSP winner	011	001	010

Table 1: The SSP winners for P' w.r.t. different orders over the issues.

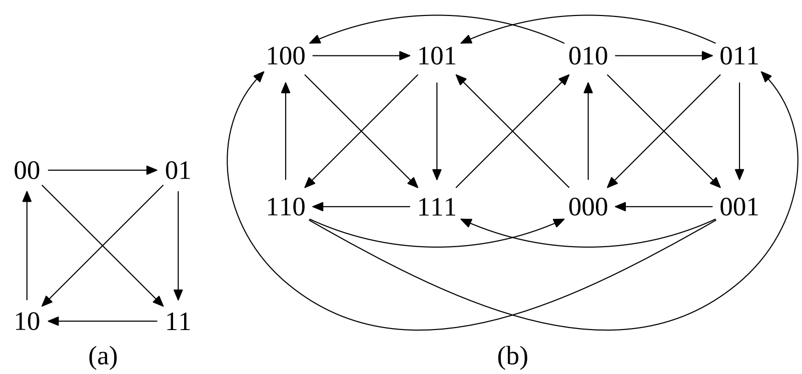

Figure 2: (a) The majority graph of the profile for $p = 2$. (b) The majority graph of the profile for $p = 3$, where four edges are not shown in the graph. They are $100 \rightarrow 000$, $101 \rightarrow 001$, $110 \rightarrow 010$, and $111 \rightarrow 011$. The directions of the other edges are defined arbitrarily. 000 is the abbreviation for $0_1 0_2 0_3$, etc.

When $p \geq 4$, $p! > 2^p$. However, it is not immediately clear whether all 2^p alternatives can be made to win by changing the order over the issues. The next theorem shows that this can actually be done.

Theorem 1 *For any $p \geq 4$ and any $n \geq 142 + 4p$, there exists an n-profile P such that for every alternative $\vec{d}$, there exists a order $\mathcal{O}'$ over $\mathcal{I}$ such that $SSP_{\mathcal{O}'}(P) = \vec{d}$.*

The proof of the theorem is by induction on p. Surprisingly, the hardest part in the inductive proof is the base case: when $p = 4$, we need to construct a profile P such that each of the 16 alternatives can be made to win in SSP by using at least one out of the 24 orders over four issues. We explicitly construct such a profile in the appendix. We prove directly that it satisfied the desired property, and we also wrote a computer program to verify its correctness. The code (written in Java) can be found at `http://www.cs.duke.edu/~lxia/Files/SSP.zip`.

4 Minimax Satisfaction Index

In the rest of this paper, we will show that strategic sequential voting on multi-issue domains is prone to paradoxes that are almost as severe as previously studied multiple-election paradoxes under models that are not game-theoretic [5, 12]. To facilitate the presentation of these results, we define an index that is intended to measure one aspect of the quality of a voting rule, called the *minimax satisfaction index*.

In words, the minimum satisfaction index is defined as follows. For each profile, consider the highest position that the winner obtains across all of the input rankings of the alternatives (corresponding to the most-satisfied voter); this is the *maximum satisfaction index* for this profile. Then, the minimax satisfaction index is obtained by taking the minimum over all profiles of the maximum satisfaction index.

Definition 2 *For any voting rule r, the* minimax satisfaction index (MSI) *of r is defined by*

$$MSI_r(m, n) = \min_{P \in L(\mathcal{X})^n} \max_{i \leq n} \big(m + 1 - rank_{V_i}(r(P))\big)$$

where m is the number of alternatives and n is the number of voters.

The MSI of a voting rule is not the final word on it. For example, the MSI for dictatorships is m, the maximum possible value, which is not to say that dictatorships are desirable. However, if the MSI of a voting rule is low, then this implies the existence of a paradox for it, namely, a profile that results in a winner that makes all voters unhappy.

Many of the multiple-election paradoxes known so far implicitly refer to such an index. For example, Lacy and Niou [12] and Benoit and Kornhauser [2] showed that for multiple referenda, if voters vote on issues separately (under some assumptions on how voters vote), then there exists a profile such that in each vote, the winner is ranked near the bottom–therefore the rule has a very low MSI.

5 Multiple-Election Paradoxes for Strategic Sequential Voting

In this section, we show that over multi-binary-issue domains, for any natural number n that is sufficiently large (we will specify the number in our theorems), there exists an n-profile P such that $SSP_{\mathcal{O}}(P)$ is ranked almost in the bottom position in each vote in P. That is, the minimax satisfaction index is extremely low for the strategic sequential voting procedure.

We first calculate the MSI for $SSP_{\mathcal{O}}$ when the winner does not depend on the tie-breaking mechanism. That is, either n is odd, or n is even and there is never a tie in any stage of running the election sequentially. This is our main multiple-election paradox result.

Theorem 2 *For any $p \in \mathbb{N}$ ($p \geq 2$) and any $n \geq 2p^2+1$, $MSI_{SSP_{\mathcal{O}}}(m,n) = \lfloor p/2+2 \rfloor$.*[4] *Moreover, in the profile P that we use to prove the upper bound, the winner $SSP_{\mathcal{O}}(P)$ is Pareto-dominated by $2^p-(p+1)p/2$ alternatives.*

Proof of Theorem 2: The upper bound on $MSI_{SSP_{\mathcal{O}}}(m,n)$ is constructive, that is, we explicitely construct a paradox.

For any n-profile $P=(V_1,\dots,V_n)$, we define the mapping $f_P:\mathcal{X}\rightarrow\mathbb{N}^n$ as follows: for any $c\in\mathcal{X}$, $f_P(c)=(h_1,\dots,h_n)$ such that for any $i\leq n$, h_i is the number of alternatives that are ranked below c in V_i. For any $l\leq p$, we denote $\mathcal{X}_l = D_l\times\cdots\times D_p$ and $\mathcal{O}_l = \mathbf{x}_l > \mathbf{x}_{l+1} > \cdots > \mathbf{x}_p$. For any vector $\vec{h}=(h_1,\dots,h_n)$ and any $l\leq p$, we say that $\vec{h}$ is *realizable* over $\mathcal{X}_l$ (through a balanced binary tree) if there exists a profile $P_l=(V_1,\dots,V_n)$ over $\mathcal{X}_l$ such that $f_{P_l}(SSP_{\mathcal{O}_l}(P_l))=\vec{h}$. We first prove the following lemma.

Lemma 1 *For any l such that $1\leq l<p$,*

$$\vec{h}_* = (\underbrace{0,\dots,0}_{\lfloor n/2\rfloor-p+l},\underbrace{1,\dots,1}_{p-l+1},\underbrace{2^{p-l+1}-1,\dots,2^{p-l+1}-1}_{\lceil n/2\rceil-1})$$

is realizable over $\mathcal{X}_l$.

Proof of Lemma 1: We prove that there exists an n-profile P_l over $\mathcal{X}_l$ such that $SSP_{\mathcal{O}_l}(P_l)=1_l\cdots 1_p$ and $\vec{h}_*$ is realized by P_l. For any $1\leq i\leq p-l+1$, we let $\vec{b}_i = 1_l\cdots 1_{p-i}0_{p+1-i}1_{p+2-i}\cdots 1_p$. That is, $\vec{b}_i$ is obtained from $1_l\cdots 1_p$ by flipping the value of $\mathbf{x}_{p+1-i}$. We obtain $P_l=(V_1,\dots,V_n)$ in the following steps.

1. Let $W_1,\dots,W_n$ be null partial orders over $\mathcal{X}_l$. That is, for any $i\leq n$, the preference relation W_i is empty.

2. For any $j\leq\lfloor n/2\rfloor-p+l$, we put $1_l\cdots 1_p$ in the bottom position in W_j; we put $\{\vec{b}_1,\dots,\vec{b}_{p-l+1}\}$ in the top positions in W_j.

3. For any j with $\lfloor n/2\rfloor+2\leq j\leq n$, we put $1_l\cdots 1_p$ in the top position of W_j, and we put $\{\vec{b}_1,\dots,\vec{b}_{p-l+1}\}$ in the positions directly below the top.

[4] If n is even, then to prove $MSI_{SSP_{\mathcal{O}}}(m,n)\geq\lfloor p/2+2\rfloor$, we restrict attention to profiles without ties.

4. For j with $\lfloor n/2 \rfloor - p + l + 1 \leq j \leq \lfloor n/2 \rfloor + 1$, we define preferences as follows. For any $i \leq p - l + 1$, in $W_{\lfloor n/2 \rfloor - p + l + i}$, we put $\vec{b}_i$ in the bottom position, $1_l \cdots 1_p$ in the second position from the bottom, and all the remaining b_j (with $j \neq i$) at the very top.

5. Finally, we complete the profile arbitrarily: for any $j \leq n$, we let V_j be an arbitrary extension of W_j.

Let $P_l = (V_1, \ldots, V_n)$. We note that for any $i \leq p - l + 1$, $\vec{b}_i$ beats any alternative in $\mathcal{X}_l \setminus \{1_l \cdots 1_p, \vec{b}_1, \ldots, \vec{b}_{p-l+1}\}$ in pairwise elections. Therefore, for any $i \leq p - l + 1$, the ith alternative that meets $1_l \cdots 1_p$ is $\vec{b}_i$, which loses to $1_l \cdots 1_p$ (just barely). It follows that $1_l \cdots 1_p$ is the winner, and it is easy to check that $f_{P_l}(1_l \cdots 1_p) = \vec{h}_*$. This completes the proof of the lemma. □

Because the majority rule is anonymous, for any permutation π over $1, \ldots, n$ and any $l < p$, if $(h_1, \ldots, h_n)$ is realizable over $\mathcal{X}_l$, then $(h_{\pi(1)}, \ldots, h_{\pi(n)})$ is also realizable over $\mathcal{X}_l$. For any $k \in \mathbb{N}$, we define $H_k = \{\vec{h} \in \{0,1\}^n : \sum_{j \leq n} h_j \geq k\}$. That is, H_k is composed of all n-dimensional binary vectors in each of which at least k components are 1. We next show a lemma to derive a realizable vector over $\mathcal{X}_{l-1}$ from two realizable vectors over $\mathcal{X}_l$.

Lemma 2 *Let $l < p$, and let $\vec{h}_1$, $\vec{h}_2$ be vectors that are realizable over $\mathcal{X}_l$. For any $\vec{h} \in H_{\lfloor n/2 \rfloor + 1}$, $\vec{h}_1 + (\vec{h}_2 + \vec{1}) \cdot \vec{h}$ is realizable over $\mathcal{X}_{l-1}$, where $\vec{1} = (1, \ldots, 1)$, and for any $\vec{a} = (a_1, \ldots, a_n)$ and any $\vec{b} = (b_1, \ldots, b_n)$, we have $\vec{a} \cdot \vec{b} = (a_1 b_1, \ldots, a_n b_n)$.*

Proof of Lemma 2: Without loss of generality, we prove the lemma for $\vec{h} = (\underbrace{0, \ldots, 0}_{\lceil n/2 \rceil - 1}, \underbrace{1, \ldots, 1}_{\lfloor n/2 \rfloor + 1})$. Let P_1, P_2 be two profiles over $\mathcal{X}_l$, each of which is composed of n votes, such that $f(P_1) = \vec{h}_1$ and $f(P_2) = \vec{h}_2$. Let $P_1 = (V_1^1, \ldots, V_n^1)$, $P_2 = (V_1^2, \ldots, V_n^2)$, $\vec{a} = SSP_{\mathcal{O}_l}(P_1)$, $\vec{b} = SSP_{\mathcal{O}_l}(P_2)$. We define a profile $P = (V_1, \ldots, V_n)$ over $\mathcal{X}_{l-1}$ as follows.

1. Let $W_1, \ldots, W_n$ be n null partial orders over $\mathcal{X}_{l-1}$.

2. For any $j \leq n$ and any $\vec{e}_1, \vec{e}_2 \in \mathcal{X}_l$, we let $(1_{l-1}, \vec{e}_1) \succ_{W_j} (1_{l-1}, \vec{e}_2)$ if $\vec{e}_1 \succ_{V_j^1} \vec{e}_2$; and we let $(0_{l-1}, \vec{e}_1) \succ_{W_j} (0_{l-1}, \vec{e}_2)$ if $\vec{e}_1 \succ_{V_j^2} \vec{e}_2$.

3. For any $\lceil n/2 \rceil \leq j \leq n$, we let $(1_{l-1}, \vec{a}) \succ_{W_j} (0_{l-1}, \vec{b})$.

4. Finally, we complete the profile arbitrarily: for any $j \leq n$, we let V_j be an (arbitrary) extension of W_j such that $(1_{l-1}, \vec{a})$ is ranked as low as possible.

We note that $(1_{l-1}, \vec{a})$ is the winner of the subtree in which $\mathbf{x}_{l-1} = 1_{l-1}$, $(0_{l-1}, \vec{b})$ is the winner of the subtree in which $\mathbf{x}_{l-1} = 0_{l_1}$, and $(1_{l-1}, \vec{a})$ beats $(0_{l-1}, \vec{b})$ in their pairwise election (because the votes from $\lceil n/2 \rceil$ to n rank $(1_{l-1}, \vec{a})$ above $(0_{l-1}, \vec{b})$). Therefore, $SSP_{\mathcal{O}_{l-1}}(P) = (1_{l-1}, \vec{a})$.

Finally, we have that $f_P((1_{l-1}, \vec{a})) = \vec{h}_1 + (\vec{h}_2 + \vec{1}) \cdot \vec{h}$. This is because $(1_{l-1}, \vec{a})$ is ranked just as low as in the profile P_1 for voters 1 through $\lceil n/2 \rceil - 1$; for any voter j with $\lceil n/2 \rceil \leq j \leq n$, additionally, $(0_{l-1}, \vec{b})$ needs to be placed below $(1_{l-1}, \vec{a})$, which implies that also, all the alternatives $(0_{l-1}, \vec{b}')$ for which j ranked $\vec{b}'$ below $\vec{b}$ in P_2 must be below $(1_{l-1}, \vec{a})$ in j's new vote in P. This completes the proof of the lemma. □

Now we are ready to prove the main part of the theorem. It suffices to prove that for any $n \geq 2p^2+1$, there exists a vector $\vec{h}_p \in \mathbb{N}^n$ such that each component of $\vec{h}_p$ is no more than $\lfloor p/2+1 \rfloor$, and $\vec{h}_p$ is realizable over $\mathcal{X}$. We first prove the theorem for the case in which n is odd. We show the construction by induction in the proof of the following lemma.

Lemma 3 *Let n be odd. For any $l' < p$ (such that l' is odd),*

$$\vec{h}_{l'} = (\underbrace{\lfloor l'/2 \rfloor, \ldots, \lfloor l'/2 \rfloor}_{\lceil n/2 \rceil - (l'^2+1)/2}, \underbrace{\lceil l'/2 \rceil, \ldots, \lceil l'/2 \rceil}_{\lfloor n/2 \rfloor + (l'^2+1)/2})$$

is realizable over $\mathcal{X}_{p-l'+1}$, and if $l' < p$, then

$$\vec{h}_{l'+1} = (\underbrace{\lfloor l'/2 \rfloor, \ldots, \lfloor l'/2 \rfloor}_{l'+1}, \underbrace{\lceil l'/2 \rceil, \ldots, \lceil l'/2 \rceil}_{n-(l'+5)(l'+1)/2}, \underbrace{\lceil l'/2 \rceil + 1, \ldots, \lceil l'/2 \rceil + 1}_{(l'+3)(l'+1)/2})$$

is realizable over $\mathcal{X}_{p-l'}$.

Proof of Lemma 3: The base case in which $l' = 1$ corresponds to a single-issue majority election over two alternatives, where $\lceil n/2 \rceil - 1$ voters vote for one alternative, and $\lfloor n/2 \rfloor + 1$ vote for the other, so that only the latter get their preferred alternative.

Now, suppose the claim holds for some $l' \leq p-2$; we next show that the claim also holds for $l'+2$. To this end, we apply Lemma 2 twice. Let $l = p - l' + 1$.

First, let $\vec{h}_* = (\underbrace{1, \ldots, 1}_{l'}, \underbrace{2^{l'}-1, \ldots, 2^{l'}-1}_{l'+1}, \underbrace{0, \ldots, 0}_{\lfloor n/2 \rfloor - l' + 1}, \underbrace{2^{l'}-1, \ldots, 2^{l'}-1}_{\lceil n/2 \rceil - l' - 2})$

By Lemma 1, $\vec{h}_*$ is realizable over $\mathcal{X}_l$ (via a permutation of the voters). Let $\vec{h} = (\underbrace{1, \ldots, 1}_{l'}, \underbrace{0, \ldots, 0}_{l'+1}, \underbrace{1, \ldots, 1}_{\lfloor n/2 \rfloor - l' + 1}, \underbrace{0, \ldots, 0}_{\lceil n/2 \rceil - l' - 2})$.

Then, by Lemma 2, $\vec{h}_{l'} + (\vec{h}_* + \vec{1}) \cdot \vec{h}$ is realizable over $\mathcal{X}_{l-1}$. We have the following calculation.

$$\vec{h}_{l'} + (\vec{h}_* + \vec{1}) \cdot \vec{h} = (\underbrace{\lceil l'/2 \rceil + 1, \ldots, \lceil l'/2 \rceil + 1}_{l'}, \underbrace{\lfloor l'/2 \rfloor, \ldots, \lfloor l'/2 \rfloor}_{l'+1}, \underbrace{\lceil l'/2 \rceil, \ldots, \lceil l'/2 \rceil}_{\lceil n/2 \rceil - (l'+3)(l'+1)/2}, \underbrace{\lceil l'/2 \rceil + 1, \ldots, \lceil l'/2 \rceil + 1}_{(l'+1)^2/2+1}, \underbrace{\lceil l'/2 \rceil, \ldots, \lceil l'/2 \rceil}_{\lfloor n/2 \rfloor - l' - 1})$$

The partition of the set of voters into these five groups uses the fact that $n \geq 2p^2+1$ implies $\lceil n/2 \rceil - (l'+3)(l'+1)/2 \geq 0$. After permuting the voters in this vector, we obtain the following vector which is realizable over $\mathcal{X}_{l-1}$:

$$\vec{h}_{l'+1} = (\underbrace{\lfloor l'/2 \rfloor, \ldots, \lfloor l'/2 \rfloor}_{l'+1}, \underbrace{\lceil l'/2 \rceil, \ldots, \lceil l'/2 \rceil}_{n-(l'+5)(l'+1)/2}, \underbrace{\lceil l'/2 \rceil + 1, \ldots, \lceil l'/2 \rceil + 1}_{(l'+3)(l'+1)/2})$$

We next let $\vec{h}' = (\underbrace{1, \ldots, 1}_{\lfloor n/2 \rfloor + 1}, \underbrace{0, \ldots, 0}_{\lceil n/2 \rceil - 1})$ and

$$\vec{h}'_* = (\underbrace{1, \ldots, 1}_{l'+1}, \underbrace{0, \ldots, 0}_{\lfloor n/2 \rfloor - l'}, \underbrace{2^{l'+1}-1, \ldots, 2^{l'+1}-1}_{\lceil n/2 \rceil - 1})$$

By Lemma 1, the latter is realizable over $\mathcal{X}_{l-1}$. Thus, by Lemma 2, $\vec{h}_{l'+1} + (\vec{h}'_* + \vec{1}) \cdot \vec{h}'$ is realizable over $\mathcal{X}_{l-2}$. Through a permutation over the voters, we obtain the desired vector:

$$\vec{h}_{l'+2} = (\underbrace{\lfloor l'/2 \rfloor + 1, \ldots, \lfloor l'/2 \rfloor + 1}_{\lceil n/2 \rceil - (l'+2)(l'+1)/2 - 1}, \underbrace{\lceil l'/2 \rceil + 1, \ldots, \lceil l'/2 \rceil + 1}_{\lfloor n/2 \rfloor + (l'+2)(l'+1)/2 + 1})$$

which is realizable over $\mathcal{X}_{l-2}$. Therefore, the claim holds for $l' + 2$. This completes the proof of the lemma. □

If p is odd, from Lemma 3 we know that the theorem is true, by setting $l' = p$. If p is even, then we first set $l' = p - 1$; then, the maximum component of $\vec{h}_{l'+1}$ is $\lceil l'/2 \rceil + 1 = \lceil (p-1)/2 \rceil + 1 = p/2 + 1$. Thus we have proved the upper bound in the theorem when n is odd.

When n is even, we have the following lemma (the proof is similar to the proof of Lemma 3, so we omitted its proof).

Lemma 4 *Let n be even. For any $l' < p$ (such that l' is odd),*

$$\vec{h}_{l'} = (\underbrace{\lfloor l'/2 \rfloor, \ldots, \lfloor l'/2 \rfloor}_{n/2 - (l'^2 - l' + 1)/2}, \underbrace{\lceil l'/2 \rceil, \ldots, \lceil l'/2 \rceil}_{n/2 + (l'^2 - l' + 1)/2})$$

is realizable over $\mathcal{X}_{p-l'+1}$, and if $l' + 1 \leq p$, then

$$\vec{h}_{l'+1} = (\underbrace{\lfloor l'/2 \rfloor, \ldots, \lfloor l'/2 \rfloor}_{l'+1}, \underbrace{\lceil l'/2 \rceil, \ldots, \lceil l'/2 \rceil}_{n-1-(l'+4)(l'+1)/2}, \underbrace{\lceil l'/2 \rceil + 1, \ldots, \lceil l'/2 \rceil + 1}_{(l'+2)(l'+1)/2+1})$$

is realizable over $\mathcal{X}_{p-l'}$.

The upper bound in the theorem when n is even follows from Lemma 4. Moreover, we note that in the step from l' to $l' + 1$ (respectively, from $l' + 1$ to $l' + 2$), no more than l' new alternatives are ranked lower than the winner in the profile that realizes $\vec{h}_{l'+1}$ (respectively, $\vec{h}_{l'+2}$). It follows that in the profile that realizes $\vec{h}_{l'+1}$ (respectively, $\vec{h}_{l'+2}$) in Lemma 3 or Lemma 4, the number of alternatives that are ranked lower than the winner by at least one voter is no more than $(l'+1)l'/2 + l' + 1 = (l'+1)(l'+2)/2$ (respectively, $(l'+2)(l'+3)/2$), which equals $(p+1)p/2$ if $l' + 1 = p$ (respectively, $(p+1)p/2$ if $l'+2 = p$). Therefore, in the profile that we use to obtain the upper bound, the winner under $SSP_{\mathcal{O}}$ is Pareto-dominated by $2^p - (p+1)p/2$ alternatives.

Finally, we show that $\lfloor p/2 + 2 \rfloor$ is a lower bound on $MSI_{SSP_{\mathcal{O}}}(m, n)$. Let P be an n-profile; let $SSP_{\mathcal{O}}(P) = \vec{a}$, and let $\vec{b}_1, \ldots, \vec{b}_p$ be the alternatives that $\vec{a}$ defeats in pairwise elections in rounds $1, \ldots, p$. It follows that in round j, more than half of the voters prefer $\vec{a}$ to $\vec{b}_j$, because we assume that there are no ties in the election. Therefore, summing over all votes, there are at least $p \times (\lfloor n/2 \rfloor + 1)$ occasions where $\vec{a}$ is preferred to one of $\vec{b}_1, \ldots, \vec{b}_p$. It follows that there exists some $V \in P$ in which $\vec{a}$ is ranked higher than at least $\lceil p \times (\lfloor n/2 \rfloor + 1)/n \rceil \geq \lfloor p/2 + 1 \rfloor$ of the alternatives $\vec{b}_1, \ldots, \vec{b}_p$. Thus $MSI_{SSP_{\mathcal{O}}}(m, n) \geq \lfloor p/2 + 2 \rfloor$.
(End of proof for Theorem 2.) □

We note that the number of alternatives is $m = 2^p$. Therefore, $\lfloor p/2 + 2 \rfloor$ is exponentially smaller than the number of alternatives, which means that there exists a profile for which every voter ranks the winner very close to the bottom. Moreover, $(p+1)p/2$ is still exponentially smaller than 2^p, which means that the winner is Pareto-dominated by almost every other alternative.

Naturally, we wish to avoid such paradoxes. One may wonder whether the paradox occurs only if the ordering of the issues is particularly unfortunate with respect to the preferences of the voters. If not, then, for example, perhaps a good approach is to randomly choose the ordering of the issues.[5] Unfortunately, our next result shows that we can construct a single profile that results in a paradox for *all* orderings of the issues. While it works for all orders, the result is otherwise somewhat weaker than Theorem 2: it does not show a Pareto-dominance result, it requires a number of voters that is at least twice the number of alternatives, the upper bound shown on the MSI is slightly higher than in Theorem 2, and unlike Theorem 2, no matching lower bound is shown.

Theorem 3 *For any $p, n \in \mathbb{N}$ (with $p \geq 2$ and $n \geq 2^{p+1}$), there exists an n-profile P such that for any order $\mathcal{O}$ over $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$, $SSP_{\mathcal{O}}(P) = 1_1 \cdots 1_p$, and any $V \in P$ ranks $1_1 \cdots 1_p$ somewhere in the bottom $p+2$ positions.*

Proof of Theorem 3: We first prove a lemma.

Lemma 5 *For any $c \in \mathcal{X}$, $\mathcal{C} \subset \mathcal{X}$ such that $c \notin \mathcal{C}$, and any $n \in \mathbb{N}$ ($n \geq 2m = 2^{p+1}$), there exists an n-profile that satisfies the following conditions. Let $F = \mathcal{X} \setminus (\mathcal{C} \cup \{c\})$.*

- *For any $c' \in \mathcal{C}$, c defeats c' in their pairwise election.*
- *For any $c' \in \mathcal{C}$ and $d \in F$, c' defeats d in their pairwise election.*
- *For any $V \in P$, c is ranked somewhere in the bottom $|\mathcal{C}| + 2$ positions.*

Proof of Lemma 5: We let $P = (V_1, \ldots, V_n)$ be the profile defined as follows. Let $F_1, \ldots, F_{\lfloor n/2 \rfloor + 1}$ be a partition of F such that for any $j \leq \lfloor n/2 \rfloor + 1$, $|F_j| \leq \lceil 2m/n \rceil = 1$. For any $j \leq \lfloor n/2 \rfloor + 1$, we let $V_j = [(F \setminus F_j) \succ c \succ \mathcal{C} \succ F_j]$. For any $\lfloor n/2 \rfloor + 2 \leq j \leq n$, we let $V_j = [\mathcal{C} \succ F \succ c]$. It is easy to check that P satisfies all conditions in the lemma. □

Now, let $c = 1_1 \cdots 1_p$ and $\mathcal{C} = \{0_1 1_2 \cdots 1_p, 1_1 0_2 1_3 \cdots 1_p, \ldots, 1_1 \cdots 1_{p-1} 0_p\}$. By Lemma 5, there exists a profile P such that c beats any alternative in $\mathcal{C}$ in pairwise elections, any alternative in $\mathcal{C}$ beats any alternative in $\mathcal{X} \setminus (\mathcal{C} \cup \{c\})$ in pairwise elections, and c is ranked somewhere in the bottom $p+2$ positions. This is the profile that we will use to prove the paradox.

Without loss of generality , we assume that $\mathcal{O} = \mathbf{x}_1 > \mathbf{x}_2 \cdots > \mathbf{x}_p$. (This is without loss of generality because all issues have been treated symmetrically so far.) c beats $1_1 \cdots 1_{p-1} 0_p$ in the first round; c will meet $1_1 \cdots 1_{p-2} 0_{p-1} 1_p$ in the next pairwise election, because
$1_1 \cdots 1_{p-2} 0_{p-1} 1_p$ beats every other alternative in that branch (they are all in F), and c will win; and so on. It follows that $c = SSP_{\mathcal{O}}(P)$. Moreover, all voters rank c in the bottom $p+2$ positions.
(End of proof for Theorem 3.) □

[5] Of course, for any ordering of the issues, there exists a profile that results in the paradoxes in Theorem 2; but this does not directly imply that there exists a single profile that works for all orderings over the issues.

6 Multiple-election paradoxes for SSP with restrictions on preferences

The paradoxes exhibited so far placed no restriction on the voters' preferences. While SSP is perfectly well defined for any preferences that the voters may have over the alternatives, we may yet wonder what happens if the voters' preferences over alternatives are restricted in a way that is natural with respect to the multi-issue structure of the setting. In particular, we may wonder if paradoxes are avoided by such restrictions. It is well known that natural restrictions on preferences sometimes lead to much more positive results in social choice and mechanism design—for example, single-peaked preferences allow for good strategy-proof mechanisms [3, 17].

In this section, we study the MSI for $SSP_{\mathcal{O}}$ for the following three cases: (1) voters' preferences are separable; (2) voters' preferences are $\mathcal{O}$-lexicographic; and (3) voters' preferences are $\mathcal{O}$-legal. For case (1), we show a mild paradox (and that this is effectively the strongest paradox that can be obtained); for case (2), we show a positive result; for case (3), we show a paradox that is nearly as bad as the unrestricted case.

Theorem 4 *For any $n \geq 2p$, when the profile is separable, the MSI for $SSP_{\mathcal{O}}$ is between $2^{\lceil p/2 \rceil}$ and $2^{\lfloor p/2 \rfloor + 1}$.*

That is, the MSI of $SSP_{\mathcal{O}}$ when votes are separable is $\Theta(\sqrt{m})$. We still have that $\lim_{m\rightarrow\infty} \Theta(\sqrt{m})/m = 0$, so in that sense this is still a paradox. However, its convergence rate to 0 is much slower than for $\Theta(\log m)/m$, which corresponds to the convergence rate for the earlier paradoxes.

Theorem 5 *For any $p \in \mathbb{N}$ ($p \geq 2$) and any $n \geq 5$, when the profile is $\mathcal{O}$-lexicographic, $MSI(SSP_{\mathcal{O}}) = 3 \cdot 2^{p-2} + 1$. Moreover, $SSP_{\mathcal{O}}(P)$ is ranked somewhere in the top 2^{p-1} positions in at least $n/2$ votes.*

Naturally $\lim_{m\rightarrow\infty}(3m/4+1)/m = 3/4$, so in that sense there is no paradox when votes are $\mathcal{O}$-lexicographic.

Under the previous two restrictions (separability and $\mathcal{O}$-lexicographicity), $SSP_{\mathcal{O}}$ coincides with $Seq(maj, \ldots, maj)$ (by Corollary 1 and Proposition 3, respectively). Therefore, Theorems 4 and 5 also apply to sequential voting rules as defined in [13]; furthermore, Theorem 4 also applies to seat-by-seat voting [2].

Finally, we study the MSI for $SSP_{\mathcal{O}}$ when the profile is $\mathcal{O}$-legal. Theorem 6 shows that it is nearly as bad as the unrestricted case (Theorem 2). The proof of Theorem 6 is the most involved proof in the paper and can be found in the appendix. The idea of the proof is similar to that of the proof for Theorem 2, but now we cannot apply Lemma 2, because $\mathcal{O}$-legality must be preserved. We start with a simpler result that shows the idea of the construction.

Claim 1 *There exists a way to break ties in $SSP_{\mathcal{O}}$ such that the following is true. Let $SSP'_{\mathcal{O}}$ be the rule corresponding to $SSP_{\mathcal{O}}$ plus the tiebreaking mechanism. For any $p \in \mathbb{N}$, there exists an $\mathcal{O}$-legal profile that consists of two votes, such that in one of the two votes, no more than $\lceil p/2 \rceil$ alternatives are ranked lower than the winner $SSP'_{\mathcal{O}}(P)$; and in the other vote, no more than $\lfloor p/2 \rfloor$ alternatives are ranked lower than $SSP'_{\mathcal{O}}(P)$.*

We emphasize that, unlike any of our other results, Claim 1 is based on a specific tie-breaking mechanism. The next theorem studies the more general and complicated case in which n can be either odd or even, and the winner does not depend on the tie-breaking mechanism. That is, there are no ties in the election. The situation is almost the same as in Theorem 2.

Theorem 6 *For any $p, n \in \mathbb{N}$ with $n \geq 2p^2+2p+1$, there exists an $\mathcal{O}$-legal profile such that in each vote, no more than $\lceil p/2 \rceil + 4$ alternatives are ranked lower than $SSP_{\mathcal{O}}(P)$. Moreover, $SSP_{\mathcal{O}}(P)$ is Pareto-dominated by at least $2^p - 4p^2$ alternatives.*

Of course, the lower bound on the MSI from Theorem 2 still applies when the profile is $\mathcal{O}$-legal, so together with Theorem 6 this proves that the MSI for $SSP_{\mathcal{O}}$ when the profile is $\mathcal{O}$-legal is $\Theta(\log m)$, just as in the unrestricted case.

7 Minimax satisfaction index of other common voting rules

So far, we have focused strictly on strategic sequential voting (SSP) in multi-issue domains (and voting trees, but only in the sense of their equivalence to strategic sequential voting). Hence, at this point, it may not be clear whether the paradoxes (or, in some cases, lack of paradoxes) that we have shown are due to the sequential, multi-issue nature of the process, or whether they are due to the strategic behavior, or whether such paradoxes are prevalent throughout voting settings.

First, let us address the question of to what extent they are due to strategic behavior. To answer this, it is most natural to compare to $Seq_{\mathcal{O}}(maj, \ldots, maj)$ ("truthful" sequential voting), which is only well defined when the profile is $\mathcal{O}$-legal. In fact, as we have already pointed out, our results for separable and $\mathcal{O}$-lexicographic profiles apply just as well to truthful sequential voting, because by Corollary 1 and Proposition 3, the strategic aspect makes no difference here. This only leaves the question of whether there is a paradox under $Seq_{\mathcal{O}}(maj, \ldots, maj)$ when the profile is $\mathcal{O}$-legal but not otherwise restricted; in this case, $Seq_{\mathcal{O}}(maj, \ldots, maj)$ is truly different from $SSP_{\mathcal{O}}$, as illustrated by Example 1. We answer this question by the following Proposition, which shows a much milder paradox.

Proposition 5 *For any $n \geq 2p$, when the profile is $\mathcal{O}$-legal, the MSI for $Seq_{\mathcal{O}}(maj, \ldots, maj)$ is between $2^{\lceil p/2 \rceil}$ and $2^{\lfloor p/2 \rfloor + 1}$.*

Having settled the effect of the strategic behavior, we next investigate the effect of the multi-issue nature of the setting. We do this by studying the MSI of common voting rules in non-combinatorial settings, where there is a single issue (but one that can take more than two values). In this context, studying strategic behavior seems intractable. By the Gibbard-Satterthwaite theorem [10, 20], without restrictions on preferences, no strategy-proof rules exist other than dictatorships and rules that exclude certain alternatives *ex ante*. Moreover, even with complete information, common voting rules have many different equilibria. Hence, we focus on studying the extent to which paradoxes occur when voters vote truthfully.

Specifically, we investigate the minimax satisfaction indices of positional scoring rules (including k-approval and Borda), plurality with runoff ($Pluo$), Copeland$_\alpha$, maximin, ranked pairs, Bucklin, STV, and (not necessarily balanced) voting trees. Of course, these rules can be applied to multi-issue domains as well as to any other domains, but they do not make use of multi-issue structure; in general, we just have a set of alternatives $C = \{c_1, \ldots, c_m\}$. Throughout the remainder of this section, we assume that $m \geq 3$, and that ties are broken in the order $c_1 \succ c_2 \succ \cdots \succ c_m$.

First, we prove several easy results.

Proposition 6 *Let $m, n \in \mathbb{N}$.*

- $MSI_{Dict}(m, n) = m$;
- *for any $k \leq m$, $MSI_{App_k}(m, n) = m + 1 - k$;*
- $MSI_{Pluo}(m, n) = m$;
- $MSI_{STV}(m, n) = m$;
- $MSI_{\mathit{Bucklin}}(m, n) \geq m/2$.

We also obtain bounds on MSI for other common voting rules mentioned in this paper.

Proposition 7 (Borda) *Let $m \in \mathbb{N}$. For any $n \in \mathbb{N}$ such that n is even, $MSI_{Borda}(m, n) = \lfloor m/2 + 1 \rfloor$; for any $n \in \mathbb{N}$ such that $n \geq m$, and n is odd, $MSI_{Borda}(m, n) = \lceil m/2 + 1 \rceil$.*

Proposition 8 (Copeland) *Let $m, n \in \mathbb{N}$. If either $0 < \alpha \leq 1$, or n is odd and $\alpha = 0$, then $MSI_{\mathit{Copeland}_\alpha}(m, n) \geq \alpha m/4$. For any $n \geq 2m$ such that n is even, $MSI_{\mathit{Copeland}_0}(m, n) = 2$.*

Proposition 9 (Maximin) *Let $m, n \in \mathbb{N}$ with $n \geq m - 1$. $MSI_{maximin}(m, n) \leq 3$.*

Proposition 10 (Ranked pairs) *Let $m, n \in \mathbb{N}$ with $n \geq \sqrt{m}$. $MSI_{rp}(m, n) \geq \sqrt{m}$.*

Proposition 11 (Voting trees) *Let T be a voting tree; let c be the alternative whose corresponding leaf is closest to the root among all leaves in T, and let its distance to the root be denoted l. If $l = 1$, then for any $n \geq 2m$, $MSI_{r_T}(m, n) = 3$; if $l \geq 2$, then for any $n \geq 2m$, $MSI_{r_T}(m, n) = \lfloor l/2 + 2 \rfloor$.*

Proposition 11 implies that among all voting trees for m alternatives, balanced voting trees have the highest MSI, which in some sense implies that balanced voting trees are the most resistant voting trees to multiple election paradoxes.

8 Conclusion and future work

Combinatorial voting settings, in which the space of all alternatives is exponential in size, constitute an important area in which techniques from computer science can be fruitfully applied. Perhaps the simplest and most natural combinatorial voting setting is that of multi-issue domains, where the space of alternatives is the Cartesian product of the local domains. In practice, common decisions on multiple issues are often reached by voting on the issues sequentially. In this paper, we considered a complete-information game-theoretic analysis of sequential voting on binary issues, which we called strategic sequential voting. Specifically, given that voters have complete information about each other's preferences and their preferences are strict, the game can be solved by a natural backward induction process (WSDSBI), which leads to a unique solution. We showed that under some conditions on the preferences, this process leads to the same outcome as the truthful sequential voting, but in general it can result in very different outcomes. We analyzed the effect of changing the order over the issues that voters vote on and showed that, in some elections, every alternative can be made to win by voting according to an appropriate order over the issues.

Most significantly, we showed that strategic sequential voting is prone to multiple-election paradoxes; to do so, we introduced a concept called minimax satisfaction index, which measures the degree to which at least one voter is made happy by the outcome of the election. We showed that the minimax satisfaction index for strategic sequential voting is exponentially small, which means that there exists a profile for which the winner is ranked almost in the bottom positions in all votes; even worse, the winner is Pareto-dominated by almost every other alternative. We showed that changing the order of the issues in sequential voting cannot completely avoid the paradoxes. These negative results indicate that the solution of the sequential game can be extremely undesirable for every voter. We also showed that multiple-election paradoxes can be avoided to some extent by restricting voters' preferences to be separable or lexicographic, but the paradoxes still exist when the voters' preferences are $\mathcal{O}$-legal.

For the sake of benchmarking our results, we also study the minimax satisfaction index for some common voting rules (under truthful voting). The results are summarized in Table 2. For a voting rule with a low (high) MSI, we can (cannot) find a paradox that is similar to the first type of multiple-election paradoxes—that is, a profile for which the winner is ranked in extremely low positions in all votes.

From this table, we may conclude that: (1) in sequential voting, the paradoxes are stronger when voting is strategic than when it is truthful, though of course this is no longer true if we are in a restricted setting where truthful and strategic voting lead to identical results (that is, when the profile is separable or lexicographic); (2) the strength of the paradoxes for sequential voting ranks somewhere in the middle, though perhaps somewhat more on the strong side, among standard social choice rules (when voters are assumed to vote truthfully).

There are many topics for future research. For example, given a profile, can we characterize the set of alternatives that win for some order over the issues?[8] Is there any criterion on the selection of the order over the issues? Perhaps more importantly, how can we get around the multiple-election paradoxes in sequential voting games? For example, Theorem 5 shows that if the voters' preferences are lexicographic, then we can avoid the paradoxes. It is not clear if there are other ways to avoid the paradoxes (paradoxes occur even if we restrict voters' preferences to be separable or $\mathcal{O}$-legal, as

[6]Additionally, there exists a profile P such that for *any* order $\mathcal{O}$ over the issues, the maximum satisfaction index of $SSP_{\mathcal{O}}$ for P is no more than $\log m + 2$ (Theorem 3).

[7]l is the minimum distance from the root to a leaf, $l \leq \log m$. If $l = 1$, then $MSI_{r_T}(m, n) = 3$.

[8]This results in a *social choice set* or *correspondence*; social choice sets have recently attracted attention from computer scientists [6].

Voting rule	MSI	
Dictatorships	m	(Proposition 6)
Plu w/ runoff	m	(Proposition 6)
STV	m	(Proposition 6)
Copeland$_\alpha$ $(0 < \alpha \leq 1)$	$\Theta(m)$	(Proposition 8)
Borda $(n \geq m)$	$\Theta(m)$	(Proposition 7)
Bucklin	$\Theta(m)$	(Proposition 6)
$Seq_{\mathcal{O}}(maj, \ldots, maj)$ ($\mathcal{O}$-lexico profiles)	$3m/4 + 1$	(Theorem 5)
$SSP_{\mathcal{O}}$ ($\mathcal{O}$-lexico profiles)	$3m/4 + 1$	(Theorem 5)
k-Approval (incl. Plurality and Veto)	$m + 1 - k$	(Proposition 6)
Ranked pairs $(n \geq \sqrt{m})$	$\Omega(\sqrt{m})$	(Proposition 10)
$Seq_{\mathcal{O}}(maj, \ldots, maj)$ (separable profiles)	$\Theta(\sqrt{m})$	(Theorem 4)
$SSP_{\mathcal{O}}$ (separable profiles)	$\Theta(\sqrt{m})$	(Theorem 4)
$Seq_{\mathcal{O}}(maj, \ldots, maj)$ ($\mathcal{O}$-legal profiles)	$\Theta(\sqrt{m})$	(Proposition 5)
$SSP_{\mathcal{O}}$ ($\mathcal{O}$-legal profiles)	between $\lfloor \log m/2 + 2 \rfloor$ and $\lfloor \log m/2 + 5 \rfloor$	(Theorem 6)
$SSP_{\mathcal{O}}$ [6]	$\lfloor \log m/2 + 2 \rfloor$	(Theorem 2)
Voting tree $(n \geq 2m)$	$\lfloor l/2 + 2 \rfloor$ [7]	(Proposition 11)
Maximin $(n \geq m - 1)$	≤ 3	(Proposition 9)
Copeland$_0$ (n is even)	2	(Proposition 8)

Table 2: The minimax satisfaction index for strategic sequential voting (SSP), truthful sequential voting (Seq), and common voting rules, ranked roughly from high to low. ("Roughly" because, for example, k-approval is really a family of voting rules, and plurality (namely, 1-approval) has a high MSI of m, whereas veto (namely, $(m-1)$-approval) has a low MSI of 2.) For multi-issue domains, $m = 2^p$, $p \in \mathbb{N}$. A low MSI implies the existence of a paradox for the rule. Results for SSP (Seq) are highlighted in dark grey (light grey).

shown in Theorem 4 and Theorem 6). Another approach is to consider other, non-sequential voting procedures for multi-issue domains. What are good examples of such procedures? Will these avoid paradoxes? What is the effect of strategic behavior for such procedures? How should we even define "strategic behavior" for such procedures, or for sequential voting with non-binary issues, or for voting rules in general? How can we extend these results to incomplete-information settings?[9] Also, beyond proving paradoxes for individual rules, is it possible to show a general impossibility result that shows that under certain minimal conditions, paradoxes cannot be avoided?[10]

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[9] Of course, because the complete-information setting is a special case of incomplete-information settings, things can only get worse in the latter.

[10] This may require quite restrictive conditions or a different notion of a paradox—for example, we have already shown that several natural voting rules have MSI m, albeit under truthful voting.

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A Proofs

We use CP-nets to better present our proofs. CP-nets [4] are a useful language for expressing preferences compactly over multi-issue domains. A CP-net $\mathcal{N}$ over $\mathcal{X}$ consists of two components: (a) a directed graph $G = (\mathcal{I}, E)$ (an edge in this graph indicates that the preferences for one issue may depend on the value of another issue) and (b) a set of conditional linear preferences $\succeq^i_{\vec{u}}$ over D_i, for any $i \leq p$ and any valuation $\vec{u}$ of the parents of $\mathbf{x}_i$ in G (denoted by $Par_G(\mathbf{x}_i)$). These conditional linear preferences $\succeq^i_{\vec{u}}$ over D_i form the *conditional preference table* for issue $\mathbf{x}_i$, denoted by $CPT(\mathbf{x}_i)$. When G is acyclic, $\mathcal{N}$ is said to be an *acyclic CP-net*. When G has no edges at all, $\mathcal{N}$ is said to be a *separable CP-net*.

Given a voter's CP-net, if we take an alternative and change the value of a single issue to obtain a new alternative, we can determine whether the voter prefers the old or the new alternative, based on the CPT for that issue. We can then make further inferences about the voter's preferences based on transitivity, although we will in general not be able to infer the voter's entire linear order over all alternatives. More precisely, a CP-net $\mathcal{N}$ induces the partial order $\succ_{\mathcal{N}}$, defined as the transitive closure of $\{(a_i, \vec{u}, \vec{z}) \succ (b_i, \vec{u}, \vec{z})) \mid i \leq p; \vec{u} \in D_{Par_G(\mathbf{x}_i)}; a_i, b_i \in D_i \text{ s.t. } a_i \succ^i_{\vec{u}} b_i; \vec{z} \in D_{\mathcal{I} \setminus (Par_G(\mathbf{x}_i) \cup \{\mathbf{x}_i\})}\}$. It is known [4] that if $\mathcal{N}$ is acyclic, then $\succ_{\mathcal{N}}$ is transitive and asymmetric, that is, a strict partial order. (This is not necessarily the case if $\mathcal{N}$ is not acyclic.)

A linear order V *extends* a CP-net $\mathcal{N}$, denoted by $V \sim \mathcal{N}$, if it extends $\succeq_{\mathcal{N}}$—that is, it is consistent with the preferences implied by the CP-net. For any valuation $\vec{u}$ of $Par_G(\mathbf{x}_i)$, let $V|_{\mathbf{x}_i:\vec{u}}$ and $\mathcal{N}|_{\mathbf{x}_i:\vec{u}}$ denote the restriction of V (or equivalently, $\mathcal{N}$) to $\mathbf{x}_i$, given $\vec{u}$. That is, $V|_{\mathbf{x}_i:\vec{u}}$ (or $\mathcal{N}|_{\mathbf{x}_i:\vec{u}}$) is the linear order $\succeq^i_{\vec{u}}$. A linear order V is *separable* if it extends a separable CP-net.

For any linear order $\mathcal{O}$ over the set of issues $\mathcal{I}$, a graph G on $\mathcal{I}$ is *compatible* with $\mathcal{O}$ if for any edge $\mathbf{x}_i \to \mathbf{x}_j$ in G, we must have that $i >_{\mathcal{O}} j$. V is *$\mathcal{O}$-legal* (or *compatible* with $\mathcal{O}$) if there exists a CP-net $\mathcal{N}$ such that V extends $\mathcal{N}$, and the graph of $\mathcal{N}$ is compatible with $\mathcal{O}$. Intuitively, this means that if we use sequential voting according to the order $\mathcal{O}$, then a voter with $\mathcal{O}$-legal preferences V will, at any point in the procedure, have well-determined preferences over the current issue, because the values for its parents have already been determined. (This is ignoring strategic considerations about later issues.) It should be noted that any separable vote is $\mathcal{O}$-legal for any ordering $\mathcal{O}$ of issues.

Because a CP-net gives only a partial order over the alternatives, it make sense to consider natural extensions of the CP-net, that is, natural ways of filling in the missing preferences. One natural way is the *lexicographic* extension, which assumes that earlier issues are always more important. Formally, the *$\mathcal{O}$-lexicographic extension* of a CP-net $\mathcal{N}$ is the linear order V that extends $\mathcal{N}$, such that for any $i \leq p$, any $a_i, b_i \in D_i$, any $\vec{u} \in D_1 \times \cdots \times D_{i-1}$, and any $\vec{z}_1, \vec{z}_2 \in D_{i+1} \times \cdots \times D_p$, we have that $(\vec{u}, a_i, \vec{z}_1) \succ_V (\vec{u}, b_i, \vec{z}_2)$ if and only if $a_i \succ_{\mathcal{N}|_{\mathbf{x}_i:\vec{u}}} b_i$. We say that a profile is $\mathcal{O}$-lexicographic if each of its votes is the lexicographic extension of a CP-net that is compatible with $\mathcal{O}$.

Proof sketch of Proposition 2. The states correspond to the local elections in which an issue is decided. Suppose that we have managed to apply WSDSBI to solve the last k stages of the game, thereby replacing the states of the $(p-k+1)$th stage with terminal states. Then, each state in the $(p-k)$th stage is a majority election between two alternatives, where each voter has a strict preference between these two alternatives. Because the rule used is majority, it is weakly dominant for each voter to vote for her preferred alternative, so we can solve the $(p-k)$th stage as well. $\square$

Proof of Proposition 3: We prove the proposition by induction on p. Suppose the proposition is true over $D_{-1} = D_2 \times \cdots \times D_p$. For any $V \in P$, because V is $\mathcal{O}$-lexicographic, we have that if $0_1 \succ_{V|_{\mathbf{x}_1}} 1_1$, then for any $\vec{a}, \vec{b} \in \mathcal{X}_2$, $(0_1, \vec{a}) \succ_V (1_1, \vec{b})$, and vice versa. Therefore, in the first round, it is a dominant strategy for every voter to truthfully submit the restriction of her preferences over $\mathbf{x}_1$; hence, by its game-theoretical interpretation, under $SSP_{\mathcal{O}}$, the first issue is set to the value $maj(P|_{\mathbf{x}_1})$. Then, by the induction hypothesis, the winner over D_{-1} is the same for both SSP and Seq. Therefore, we have that $SSP_{\mathcal{O}}(P) = Seq(maj, \ldots, maj)(P)$. $\square$

Proof of Proposition 4: We prove the proposition by induction on p. Suppose the proposition is true over $D_{-p} = D_1 \times \cdots \times D_{p-1}$. We consider the winners in the bottom layer of the voting tree corresponding to SSP. For any $\vec{a} \in D_{-p}$, the voters are comparing $(\vec{a}, 0_p)$ and $(\vec{a}, 1_p)$. Because P is $inv(\mathcal{O})$-legal, for any $\vec{a} \in D_{-p}$ and any $j \leq n$, voter j's preferences over $\mathbf{x}_p$ are independent of the other issues. Thus, she prefers $(\vec{a}, 0_p)$ to $(\vec{a}, 1_p)$ if and only if $0_p \succ_{V_j|_{\mathbf{x}_p}} 1_p$. Therefore, the winning value for $\mathbf{x}_p$ is $d_p = maj(P|_{\mathbf{x}_p})$ *everywhere* in the first round of the voting tree, and the corresponding alternatives propagate up to the next level in the tree. For these remaining alternatives, we only need the restricted preference profile $P|_{\mathbf{x}_{-p}:d_p}$, which is $inv(\mathcal{O}_{-p})$-legal (where $inv(\mathcal{O}_{-p})$ is the order $\mathbf{x}_{p-1} > \cdots > \mathbf{x}_1$). By the induction hypothesis, the winner for the rest of the voting tree is the same as $Seq_{inv(\mathcal{O}_{-p})}(maj, \ldots, maj)(P|_{\mathbf{x}_{-p}:d_p})$. It follows that $SSP_{\mathcal{O}}(P) = Seq_{inv(\mathcal{O})}(maj, \ldots, maj)(P)$. $\square$

Proof of Theorem 1: We prove the theorem by induction on the nubmer of issues p. We first show how to construct a desirable majority graph $\mathcal{M}$ for $p = 4$, then we show how to construct a n-profile that corresponds to $\mathcal{M}$.

To define $\mathcal{M}$, we first define a majority graph $\mathcal{M}_3$ over $\mathcal{X}_3 = D_2 \times D_3 \times D_4$, as in Figure 3 plus the following four edges: $100 \to 000$, $101 \to 001$, $110 \to 010$, and $111 \to 011$. This graph is essentially the same as Figure 2(b) in Example 2. Note that $\mathcal{M}_3$ is defined over $D_2 \times D_3 \times D_4$ and Figure 2(b) is defined over $D_1 \times D_2 \times D_3$. For any $\vec{a} = (a_2, a_3, a_4) \in \mathcal{X}_3$, let $f(\vec{a}) = (1_1, \vec{a})$ and let $g(\vec{a}) = (0_1, \bar{a}_2, a_3, a_4)$. For example, $f(0_2 0_3 0_4) = 1_1 0_2 0_3 0_4$ and $g(0_2 0_3 0_4) = 1_1 1_2 0_3 0_4$. We define $\mathcal{M}$ as follows.

(1) The subgraph of $\mathcal{M}$ over $\{1_1\} \times \mathcal{X}_3$ is $f(\mathcal{M}_3)$. That is, for any $\vec{a}, \vec{b} \in \mathcal{X}_3$, if $\vec{a} \to \vec{b}$ in $\mathcal{M}'$, then $f(\vec{a}) \to f(\vec{b})$ in $\mathcal{M}$.

(2) The subgraph of $\mathcal{M}$ over $\{0_1\} \times \mathcal{X}_3$ is $g(\mathcal{M}_3)$.

(3) For any $\vec{a} \in \mathcal{X}_3$, we have $(1_1, \vec{a}) \to (0_1, \vec{a})$. For any $\vec{a} \in \mathcal{X}_3$ and $\vec{a} \neq 111$, we have $g(\vec{a}) \to f(\vec{a})$.

(4) We then add the following edges to $\mathcal{M}$. $0100 \to 1110$, $1000 \to 0010$, $1101 \to 0111$, $0001 \to 1011$, $1101 \to 0100$, $1000 \to 0001$, $0001 \to 1101$, $0100 \to 1000$, $1111 \to 0110$, $1100 \to 0101$, $0011 \to 1010$, $1001 \to 0000$, $1111 \to 0011$, $0011 \to 1100$, $0011 \to 1001$, $1111 \to 0000$.

(5) Any other edge that is not defined above is defined arbitrarily.

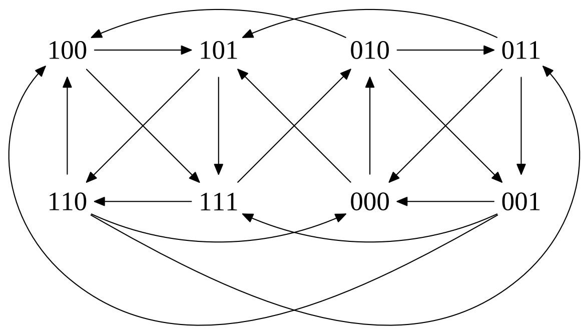

Figure 3: $\mathcal{M}_3$, where four edges are not shown in the figure. They are $100 \to 000$, $101 \to 001$, $110 \to 010$, and $111 \to 011$. The directions of the other edges are defined arbitrarily. 000 is the abbreviation for $0_2 0_3 0_4$, etc.

Let P be an arbitrary profile whose majority graph satisfies conditions (1) through (4) above. We make the following observations.

- If $\mathbf{x}_1$ is the first issue in $\mathcal{O}'$, then the first component of SSP$_{\mathcal{O}'}(P)$ is 1_1. Moreover, every alternative whose first component is 1_1 (except 1111 and 1000) can be made to win by changing the order of $\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$.
- If $\mathbf{x}_1$ is the last issue in $\mathcal{O}'$, then the first component of SSP$_{\mathcal{O}'}(P)$ is 0_1. Moreover, every alternative whose first component is 0_1 (except 0011 and 0100) can be made to win by changing the order of $\mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4$.
- Let $\mathcal{O}' = \mathbf{x}_3 > \mathbf{x}_1 > \mathbf{x}_2 > \mathbf{x}_4$, we have SSP$_{\mathcal{O}'}(P) = 0100$; let $\mathcal{O}' = \mathbf{x}_3 > \mathbf{x}_1 > \mathbf{x}_4 > \mathbf{x}_2$, we have SSP$_{\mathcal{O}'}(P) = 1000$; let $\mathcal{O}' = \mathbf{x}_4 > \mathbf{x}_1 > \mathbf{x}_3 > \mathbf{x}_2$, we have SSP$_{\mathcal{O}'}(P) = 0011$; let $\mathcal{O}' = \mathbf{x}_2 > \mathbf{x}_4 > \mathbf{x}_1 > \mathbf{x}_3$, we have SSP$_{\mathcal{O}'}(P) = 1111$.

In summary, every alternative is a winner of SSP w.r.t. at least one order over the issues. The reader can also check out the java program online at `http://www.cs.duke.edu/~lxia/Files/SSP.zip`, to verify the correctness of such a construction. We notice that conditions (1) through (4) imposes 79 constraints on pairwise comparisons. Therefore, usinging McGarvey's trick [14], for any $n \geq 2 \times 79 = 158$, there exists an n-profile whose majority graph satisfies conditions (1) through (4). This means that the theorem holds for $p = 4$.

Now, suppose that the thoerem holds for $p = p'$. Let $P = (V_1, \ldots, V_n)$ be an n-profile over $\mathcal{X}' = D_2 \times \ldots \times D_{p'+1}$ such that $n \geq 142 + 4p'$ and each alternative in $\mathcal{X}'$ can be made to win in SSP by changing the order over $\mathbf{x}_2, \ldots, \mathbf{x}_{p'+1}$. Let $\mathcal{X} = D_1 \times \cdots \times D_{p'+1}$. Let $f : \mathcal{X}' \to \mathcal{X}$ be the mapping defined as follows. For any $\vec{a} \in \mathcal{X}'$, $f(\vec{a}) = (1_1, \vec{a})$. That is, for any $\vec{a} \in \mathcal{X}'$, f concatenates 1_1 and $\vec{a}$. Let $g : \mathcal{X}' \to \mathcal{X}$ be the mapping defined as follows. For any $\vec{a} = (a_2, \ldots, a_{p'+1}) \in \mathcal{X}'$, $g(\vec{a}) = (0_1, \bar{a}_2, \ldots, \bar{a}_{p'+1})$. That is, for any $\vec{a} \in \mathcal{X}'$, g flips the first two components of $f(\vec{a})$. Next, we define an $(n+4)$-profile $P' = (V'_1, \ldots, V'_{n+4})$ as follows.

For any $i \leq 2\lfloor (n-1)/2 \rfloor$, we let $V'_i = \begin{cases} f(V_i) \succ g(V_i) & \text{if } i \text{ is odd} \\ g(V_i) \succ f(V_i) & \text{if } i \text{ is even} \end{cases}$. For any

$2\lfloor (n-1)/2 \rfloor + 1 \leq i \leq n$, we let $V'_i = [f(V_i) \succ g(V_i)]$. For any $j \leq 4$, we let

$$V'_{n+j} = \begin{cases} g(0_2 \dots 0_{p+1}) \succ f(0_2 \dots 0_{p+1}) \succ g(0_2 \dots 0_p 1_{p+1}) \\ \succ f(0_2 \dots 0_p 1_{p+1}) \succ g(1_2 \dots 1_{p+1}) \succ f(1_2 \dots 1_{p+1}) & \text{if } j \text{ is odd} \\ g(1_2 \dots 1_{p+1}) \succ f(1_2 \dots 1_{p+1}) \succ g(1_2 \dots 1_p 0_{p+1}) \\ \succ f(1_2 \dots 1_p 0_{p+1}) \succ g(0_2 \dots 0_{p+1}) \succ f(0_2 \dots 0_{p+1}) & \text{if } j \text{ is even} \end{cases}$$

For any pair of alternatives c, c', and any profile P^*, we let $D_{P^*}(c, c')$ denote the number of times that c is preferred to c', minus the number of times c' is preferred to c, both in the profile P^*. That is, $D_{P^*}(c, c') > 0$ if and only if c beats c' in their pairwise election. We make the following observations on P'.

- For any $\vec{a} \in \mathcal{X}'$, $D_{P'}(f(\vec{a}), g(\vec{a})) > 0$ and $D_{P'}((1_1, \vec{a}), (0_1, \vec{a})) > 0$.
- For any $\vec{a}, \vec{b} \in \mathcal{X}'$ (with $\vec{a} \neq \vec{b}$), $D_{P'}(f(\vec{a}), f(\vec{b})) > 0$ if and only if $D_P(\vec{a}, \vec{b}) > 0$; $D_{P'}(g(\vec{a}), g(\vec{b})) > 0$ if and only if $D_P(\vec{a}, \vec{b}) > 0$.

It follows that for any order $\mathcal{O}'$ over $\{\mathbf{x}_2, \dots, \mathbf{x}_{p+1}\}$, we have $\text{SSP}_{[\mathbf{x}_1 \succ \mathcal{O}']}(P') = f(SSP_{\mathcal{O}'}(P))$ (because after voting on issue $\mathbf{x}_1$, all alternatives whose first component is 0_1 are eliminated, then it reduces to SSP over $\mathcal{X}'$); we also have that $\text{SSP}_{[\mathcal{O}' \succ \mathbf{x}_1]}(P') = g(SSP_{\mathcal{O}'}(P))$ (because in the last round, the two competing alternatives are considering are $f(SSP_{\mathcal{O}'}(P))$ and $g(SSP_{\mathcal{O}'}(P))$, and the majority of voters prefer the latter). We recall that each alternative in $\mathcal{X}'$ can be made to win w.r.t. an order O' over $\{\mathbf{x}_2, \dots, \mathbf{x}_{p'+1}\}$. It follows that each alternative in $\mathcal{X}$ can also be made to win w.r.t. an order over $\{\mathbf{x}_1, \dots, \mathbf{x}_{p'+1}\}$, which means that the theorem holds for $p = p' + 1$. Therefore, the theorem holds for any $p \geq 4$. $\square$

Proof of Theorem 4: Let $P = (V_1, \dots, V_n)$. For any $i \leq p$, we let $d_i = maj(P|_{\mathbf{x}_i})$. That is, d_i is the majority winner for the projection of the profile to the ith issue. Because any separable profile is compatible with any order over the issues, P is a $\mathcal{O}^{-1}$-legal profile. It follows from Corollary 1 that $SSP_{\mathcal{O}}(P) = (d_1, \dots, d_p)$. Without loss of generality $(d_1, \dots, d_p) = (1_1, \dots, 1_p)$.

First, we prove the lower bound. Because for any $i \leq p$, at least half of the voters prefer 1_i to 0_i, we have that the total number of times that a voter prefers 1 to 0 for an issue, counted across all voters and issues, is at least $p \cdot (n/2)$. Therefore, there exists $j \leq n$ such that voter j prefers 1 to 0 on at least $p/2$ issues, otherwise the total number of times that a voter prefers 1 to 0 for an issue, counted across all voters and issues, is no more than $n \cdot (p/2) - 1 < p \cdot (n/2)$, which is a contradiction. Formally put, there exists $j \leq n$ such that $|\{i \leq p : 1_i \succ_{V_j} 0_i\}| \geq p/2$. Without loss of generality for any $i \leq \lceil p/2 \rceil$, $1_i \succ_{V_j} 0_i$. It follows that for any $\vec{a} \in D_1 \times \cdots D_{\lceil p/2 \rceil}$, we have that $(1_1, \dots, 1_p) \succ_{V_j} (\vec{a}, 1_{\lceil p/2 \rceil + 1}, \dots, 1_p)$. Therefore, the minimax satisfaction index is at least $2^{\lceil p/2 \rceil}$.

Next, we prove the upper bound. We show that there exists a set of n CP-nets $\mathcal{N}_1, \dots, \mathcal{N}_n$ such that for any $j \leq n$, the number of issues on which $\mathcal{N}_j$ prefers 1 to 0 is exactly $\lfloor p/2 \rfloor + 1$, and for any $i \leq p$, $maj(\mathcal{N}_1|_{\mathbf{x}_i}, \dots, \mathcal{N}_n|_{\mathbf{x}_i}) = 1_i$. The proof is by explicitly constructing the profile through the following n-step process. Let $k_1 = \cdots = k_p = \lfloor p/2 \rfloor + 1$. In the jth step, we let $I_j = \{i_1, \dots, i_{\lfloor p/2 \rfloor + 1}\}$ be the set of indices of the highest k's. Then, for any $i \in I_j$, we let $\mathcal{N}_j|_{\mathbf{x}_i} = [1_i \succ 0_i]$ and $k_i \leftarrow k_i - 1$; for any $i \notin I_j$, $\mathcal{N}_j|_{\mathbf{x}_i} = [0_i \succ 1_i]$. Because $n \geq 2p$, we have that $n(\lfloor p/2 \rfloor + 1) \geq p(\lfloor n/2 \rfloor + 1)$, which means that after n steps all $k_i \leq 0$ for all $i \leq p$. It follows that $\mathcal{N}_1, \ldots, \mathcal{N}_n$ satisfy the properties we are looking for. Now, let $V_1, \ldots, V_n$ be extensions of $\mathcal{N}_1, \ldots, \mathcal{N}_n$, respectively, such that for any $j \leq n$, $1_1 \cdots 1_p$ is ranked as low as possible in any V_j. We have the following observation on the number of alternatives that are ranked lower than $1_1 \cdots 1_p$ in $V_1, \ldots, V_n$.

Lemma 6 *For any partial order W and any alternative c, we let $|Down_W(c)| = \{\vec{c}' : c \succeq_W c'\}$, that is, $|Down_W(c)|$ is the set of all alternatives (including c) that are less preferred to c in W. There exists a linear order V such that V extends W and c is ranked in the $|Down_W(c)|$th position from the bottom.*

We note that for any $j \leq p$, $|Down_{\succ_{\mathcal{N}_j}}(1_1 \cdots 1_p)| = 2^{\lfloor p/2+1 \rfloor}$ (we remember that $\succ_{\mathcal{N}_j}$ is the partial order that $\mathcal{N}_j$ encodes). It follows from Lemma 6 that for any $j \leq n$, $1_1 \cdots 1_p$ is ranked in the $2^{\lfloor p/2+1 \rfloor}$th position from the bottom in V_j.

(End of proof for Theorem 4.) □

Proof of Theorem 5: The proof is for profiles without ties. Without loss of generality $SSP_{\mathcal{O}}(P) = 1_1 \cdots 1_p$ and for any $j \leq \lfloor n/2 \rfloor + 1$, $1_1 \succ_{V_j|_{\mathbf{x}_1}} 0_1$. It follows that in $V_1, \ldots, V_{\lfloor n/2 \rfloor + 1}$, $1_1 \cdots 1_p$ is ranked within top $m/2$ positions. Because in at least $\lfloor n/2 \rfloor + 1$ votes $1_1 : 1_2 \succ 0_2$, there exists a vote $V \in P$ such that $1_1 \succ_{V|_{\mathbf{x}_1}} 0_1$ and $1_1 : 1_2 \succ_{V|_{\mathbf{x}_2:1_1}} 0_2$. It follows that $1_1 \cdots 1_p$ is ranked in the bottom $(3 \cdot 2^{p-2} + 1)$th position. This proves that when the profile is $\mathcal{O}$-lexicographic, $MSI(SSP_{\mathcal{O}}) \geq 3 \cdot 2^{p-2} + 1$.

We next prove that $3 \cdot 2^{p-2} + 1$ is also an upper bound. We consider the following profile $P = (V_1, \ldots, V_n)$, defined as follows. For any $j \leq \lfloor n/2 \rfloor + 1$, $1_1 \succ_{V_j|_{\mathbf{x}_1}} 0_1$; for $j = 1, 2$, $1_2 \succ_{V_j|_{\mathbf{x}_2:1_1}} 0_2$; for any $2 \leq i \leq p$, $1_i \succ_{V_i|_{\mathbf{x}_i:1_1 \cdots 1_{i-1}}} 0_j$, and for any $3 \leq j \leq n$, $1_i \succ_{V_j|_{\mathbf{x}_i:1_1 \cdots 1_{i-1}}} 0_j$; for any local preferences of any voter that is not defined above, let 0 be preferred to 1.

We note that for any $i \leq p$, more than $n/2$ votes in $P|_{\mathbf{x}_i:1_1 \cdots 1_{i-1}}$ prefers 1_i to 0_i, which means that $SSP_{\mathcal{O}}(P) = 1_1 \cdots 1_p$. It is easy to check that in any vote, $1_1 \cdots 1_p$ is ranked somewhere within bottom $3 \cdot 2^{p-2} + 1$ positions. □

Proof of Claim 1: The proof is by induction on p. When $p = 2$, let the CPT of $\mathcal{N}_1$ be $0_1 \succ 1_1, 0_1 : 1_2 \succ 0_2, 1_1 : 1_2 \succ 0_2$; let the CPT of $\mathcal{N}_2$ be $1_1 \succ 0_1, 0_1 : 0_2 \succ 1_2, 1_1 : 0_2 \succ 1_2$; $V_1 = [0_1 1_2 \succ 0_1 0_2 \succ 1_1 1_2 \succ 1_1 0_2]$; $V_2 = [1_1 0_2 \succ 0_1 0_2 \succ 1_1 1_2 \succ 0_1 1_2]$. In the first step, ties are broken in favor of $1_1 1_2$. Given 1_1, ties are broken in favor of 1_2; given 0_1, ties are broken in favor of 1_2.

Suppose the claim is true for $p = l$. Next we construct $\mathcal{N}_1$ and $\mathcal{N}_2$ for $p = l+1$. Let $\mathcal{N}_1'$, $\mathcal{N}_2'$, V_1', V_2' be the CP-nets and the votes for the case of $p = l$, where the multi-issue domain is $D_2 \times \cdots \times D_{l+1}$. Without loss of generality $|Down_{V_1}(1_2 \cdots 1_{l+1})| \leq \lceil l/2 \rceil$ and $|Down_{V_2}(1_2 \cdots 1_{l+1})| \leq \lfloor l/2 \rfloor$. Let $\vec{e} \in D_2 \times \cdots \times D_{l+1}$ be an arbitrary alternative such that $1_2 \cdots 1_{l+1} \succ_{V_2} \vec{e}$. Such $\vec{e}$ always exists, because if on the contrary $1_2 \cdots 1_{l+1}$ is in the bottom of V_2, it must be ranked higher than at least l other alternatives in V_1 to win the election, which contradicts the assumption that $|Down_{V_1}(1_2 \cdots 1_{l+1})| \leq \lceil l/2 \rceil$. We will explain later why we choose $\vec{e}$ in such a way.

Let $\mathcal{N}_1^*$ (respectively, $\mathcal{N}_2^*$) be two separable CP-nets (we remember that a CP-net is separable if its graph has no edges) $D_2 \times \cdots \times D_{l+1}$ in which $\vec{e}$ is in the top (respectively, bottom) position. For $i = 1, 2$, we let $\mathcal{N}_i$ be a CP-net over $D_1 \times \cdots \times D_{l+1}$, defined as follows:

- $0_1 \succ_{\mathcal{N}_i} 1_1$.
- The sub-CP-net of $\mathcal{N}_i$ restricted on $\mathbf{x}_1 = 1_1$ is $\mathcal{N}'_i$;
- The sub-CP-net of $\mathcal{N}_i$ restricted on $\mathbf{x}_1 = 0_1$ is $\mathcal{N}^*_i$;

Let V_1, V_2 be the extension of $\mathcal{N}_1$ and $\mathcal{N}_2$ respectively, that satisfy the following conditions:

- For any $\vec{b}, \vec{d} \in D_2 \times \cdots \times D_{l+1}$ such that $\vec{b} \neq \vec{e}$, and any $i = 1, 2$, we have that $(0_1, \vec{b}) \succ_{V_i} (1_1, \vec{d})$. (This condition can be satisfied, because we have $0_1 \succ_{\mathcal{N}_i} 1_1$.)
- For any $\vec{b}, \vec{d} \in D_2 \times \cdots \times D_{l+1}$, and any $i = 1, 2$, we have that $(1_1, \vec{b}) \succ_{V_i} (1_1, \vec{d})$ if and only if $\vec{b} \succ_{V'_i} \vec{d}$. (This condition says that if we focus on the order of the alternatives whose $\mathbf{x}_1$ component is 1_1 in V_i, then it is the same as in V'_i.)
- For any $\vec{d} \in D_2 \times \cdots \times D_{l+1}$, we have that $(0_1, \vec{e}) \succ_{V_1} (1_1, \vec{d})$.
- $(1_1, \ldots, 1_{l+1}) \succ_{V_2} (0_1, \vec{e}) \succ_{V_2} (1_1, \vec{e})$.

We let the tie-breaking mechanism be as follows: in the first step, ties are broken in favor of 1_1; in the subgame in which $\mathbf{x}_1 = 1_1$, ties are broken in the same way as for the profile (V'_1, V'_2) (such that $1_2 \cdots 1_{l_1}$ is the winner for the profile); in the subgame in which $\mathbf{x}_1 = 0_1$, ties are broken in such a way that $\vec{e}$ is the winner (because $\vec{e}$ is ranked in the top position in one vote, and in the bottom position in the other, there exists a tie-breaking mechanism under which $\vec{e}$ is the winner).

We note that $1_1 \cdots 1_p \succ_{V_1} \vec{d}$ if and only if $\vec{d} = (1_1, \vec{d'})$ for some $\vec{d'} \in D_2 \times \cdots \times D_{l+1}$ such that $1_2 \cdots 1_p \succ_{V'_1} \vec{d'}$. It follows that $|Down_{V_1}(1_1 \cdots 1_{l+1})| = |Down_{V'_1}(1_2 \cdots 1_{l+1})|$. We also note that $1_1 \cdots 1_{l+1} \succ_{V_2} \vec{b}$ if and only if $\vec{b} = (0_1, \vec{e})$ or $\vec{b} = (1_1, \vec{b'})$ for some $\vec{b'} \in D_2 \times \cdots \times D_{l+1}$ such that $1_2 \cdots 1_p \succ_{V'_2} \vec{b'}$. It follows that $|Down_{V_2}(1_1 \cdots 1_{l+1})| = |Down_{V'_2}(1_2 \cdots 1_{l+1})| + 1$. Therefore, $|Down_{V_1}(1_1 \cdots 1_{l+1})| \leq \lfloor (l+1)/2 \rfloor$ and
$|Down_{V_2}(1_1 \cdots 1_p)| \leq \lfloor l/2 \rfloor + 1 \leq \lceil (l+1)/2 \rceil$.

Here the trick to choose $\vec{e}$ such that $1_2 \cdots 1_{l+1} \succ_{V'_2} \vec{e}$ is crucial, because we force $0_1 \succ_{\mathcal{N}_2} 1_1$ and $1_1 \cdots 1_{l+1} \succ_{V_2} (0_1, \vec{e})$, which implies that $1_1 \cdots 1_{l+1} \succ_{V_2} (0_1, \vec{e}) \succ_{V_2} (1_1, \vec{e})$ (since V_2 extends $\mathcal{N}_2$). If we chose $\vec{e}$ such that $\vec{e} \succ_{V'_2} 1_2 \cdots 1_{l+1}$, then we would have that $|Down_{V_2}(1_1 \cdots 1_{l+1})| = |Down_{V'_2}(1_2 \cdots 1_{l+1})| + 2$, which does not prove the claim for $p = l + 1$.

Next, we verify that $SSP_{\mathcal{O}}(V_1, V_2) = 1_1 \cdots 1_{l+1}$. We note that $(0_1, \vec{e}) \succ_{V_1} 1_1 \cdots 1_{l+1}$. Therefore, in the first step voter 1 will vote for 0_1. Meanwhile, $1_1 \cdots 1_{l+1} \succ_{V_2} (0_1, \vec{e})$, which means that in the first step voter 2 will vote for 1_1. Because ties are broken in favor of 1_1 in the first step, we will fix $\mathbf{x}_1 = 1_1$. Then, in the following steps (step $2, \ldots, l+1$), $1_2, \ldots, 1_{l+1}$ will be the winners by induction hypothesis, which means that $SSP_{\mathcal{O}}(V_1, V_2) = 1_1 \cdots 1_{l+1}$.

Therefore, the claim is true for $p = l + 1$. This means that the claim is true for any $p \in \mathbb{N}$.

Example 3 *Let us show an example of the above construction from $p = 2$ to $p = 3$. In $\mathcal{N}_1$, we have $0_1 \succ 1_1$, $1_1 : \mathcal{N}_1^*$, and $0_1 : \mathcal{N}_1'$, where $\mathcal{N}_1'$ is $0_2 \succ 1_2, 0_2 : 1_3 \succ 0_3, 1_2 : 1_3 \succ 0_3$. (We note that $\mathcal{N}_1'$ is defined over $D_2 \times D_3$.) V_1 restricted to 1_1 is $V_1' = [0_2 1_3 \succ 0_2 0_3 \succ 1_2 1_3 \succ 1_2 0_3]$ (which is, again, over $D_2 \times D_3$). Let $\vec{e} = 0_2 1_3$. Therefore, we have the following construction:*

$V_1 = 0_1 0_2 1_3 \succ 0_1 1_2 1_3 \succ 0_1 0_2 0_3 \succ 0_1 1_2 0_3 \succ 1_1 0_2 1_3 \succ 1_1 0_2 0_3 \succ 1_1 1_2 1_3 \succ 1_1 1_2 0_3$

$V_2 = 0_1 1_2 0_3 \succ 0_1 0_2 0_3 \succ 0_1 1_2 1_3 \succ 1_1 1_2 0_3 \succ 1_1 0_2 0_3 \succ 1_1 1_2 1_3 \succ 0_1 0_2 1_3 \succ 1_1 0_2 1_3$

Ties are broken in a way such that if we are in the branch in which $\mathbf{x}_1 = 1_1$, then $1_2 1_3$ is the winner; and if we are in the branch in which $\mathbf{x}_1 = 0_1$, then $\vec{e} = 0_2 1_3$ is the winner. In the first step, ties are broken in favor of 1_1. Then, the sub-game winners are $1_1 1_2 1_3$ and $0_1 0_2 1_3$. Since exactly one vote (V_1) prefers $0_1 0_2 1_3$ to $1_1 1_2 1_3$, and the other vote V_2 prefers $1_1 1_2 1_3$ to $0_1 0_2 1_3$, the winner is $1_1 1_2 1_3$.

(End of proof for Claim 1.) □

Proof of Theorem 6: For simplicity, we prove the theorem for the case in which $n = 2p^2 + 2p + 1$. The proof for the case in which $n > 2p^2 + 2p + 1$ is similar. For any $l \leq p$, we let $\mathcal{X}_l = \{0_l, 1_l\} \times \{0_{l+1}, 1_{l+1}\} \times \cdots \times \{0_p, 1_p\}$; let $\mathcal{O}_l = \mathbf{x}_l > \mathbf{x}_{l+1} > \cdots > \mathbf{x}_p$. We first prove the following claim by induction.

Claim 2 *For any $l \leq p$, there exists a $\mathcal{O}_l$-legal profile $P_l = A_l \cup B_l \cup \hat{A}_l \cup \hat{B}_l \cup \{c^l\}$ over $\mathcal{X}_l$, where $A_l = \{a_1^l, \ldots, a_{p^2}^l\}$, $B_l = \{b_1^l, \ldots, b_{p^2}^l\}$, $\hat{A}_l = \{\hat{a}_1^l, \ldots, \hat{a}_p^l\}$, $\hat{B}_l = \{\hat{b}_1^l, \ldots, \hat{b}_p^l\}$, that satisfies the following conditions.*

- *$SSP_{\mathcal{O}_l}(P_l) = 1_l \cdots 1_p$.*
- *For any $V \in P_l$, $|Down_V(1_l \cdots 1_p)| \leq \lceil (p - l + 1)/2 \rceil + 5$.*
- *For any $(p - l)p \leq j \leq p^2$, $|Down_{a_j^l}(1_l \cdots 1_p)| \leq \lceil (p - l + 1)/2 \rceil + 3$, $|Down_{b_j^l}(1_l \cdots 1_p)| \leq \lceil (p - l + 1)/2 \rceil + 3$.*
- *For any $p-l \leq j \leq p$, $|Down_{\hat{a}_j^l}(1_l \cdots 1_p)| \leq \lceil (p-l+1)/2 \rceil + 3$, $|Down_{\hat{b}_j^l}(1_l \cdots 1_p)| \leq \lceil (p - l + 1)/2 \rceil + 3$.*
- *If $p - l + 1$ is odd, then for any $V_B \in B$, $|Down_{V_B}(1_l \cdots 1_p)| \leq \lceil (p - l + 1)/2 \rceil + 4$, for any $(p - l)p \leq j \leq p^2$, $|Down_{b_j^l}(1_l \cdots 1_p)| \leq \lceil (p - l + 1)/2 \rceil + 2$, and for any $p - l \leq j \leq p$, $|Down_{\hat{b}_j^l}(1_l \cdots 1_p)| \leq \lceil (p - l + 1)/2 \rceil + 2$.*
- *For any $V \in P_l$, we have $1_l \cdots 1_p \succ_V 1_l \cdots 1_{p-2} 0_{p-1} 0_p$.*

Proof of Claim 2: We prove the claim by induction on l. When $l = p - 1$, we let all votes in P_{p-1} be

$$1_{p-1} 1_p \succ 1_{p-1} 0_p \succ 0_{p-1} 1_p \succ 0_{p-1} 0_p$$

It is easy to check that P_{p-1} satisfies all the conditions in the claim. Suppose the claim is true for $l \leq p$, we next prove that the claim is also true for $l-1$. We show the existence of P_{l-1} by construction for the following two cases.

Case 1: $p-l+1$ is even.

We let $\mathcal{N}_1^l, \ldots, \mathcal{N}_{p-l+1}^l, \mathcal{N}_A^l, \mathcal{N}_B^l$ be separable CP-nets over $\mathcal{X}_l$, defined as follows.

- Let $1_l \cdots 1_{p-2} 0_{p-1} 0_p$ be in the bottom position of $\mathcal{N}_A^l$; let $1_l \cdots 1_{p-2} 0_{p-1} 0_p$ be in the top position of $\mathcal{N}_B^l$.
- For any $1 \leq i \leq p-l-1$, let

$$1_l \cdots 1_{l+i-2} 0_{l+i-1} 1_{l+i} \cdots 1_{p-2} 0_{p-1} 0_p$$

  be in the top position of $\mathcal{N}_i^l$; let $1_l \cdots 1_{p-2} 1_{p-1} 0_p$ be in the top position of $\mathcal{N}_{p-l}^l$; let $1_l \cdots 1_{p-2} 0_{p-1} 1_p$ be in the top position of $\mathcal{N}_{p-l+1}^l$.

For any linear order V over $\mathcal{X}_l$, we let the *composition* of V and $\mathcal{N}$ (where $\mathcal{N} \in \{\mathcal{N}_1^l, \ldots, \mathcal{N}_{p-l+1}^l, \mathcal{N}_A^l, \mathcal{N}_B^l\}$) be a partial order O^{l-1} over $\mathcal{X}_{l-1}$, defined as follows.

- For any $\vec{d}_1, \vec{d}_2 \in \mathcal{X}_l$ such that $\vec{d}_1 \succ_V \vec{d}_2$, we let $(1_{l-1}, \vec{d}_1) \succ_{O^{l-1}} (1_{l-1}, \vec{d}_2)$.
- For any $\vec{d}_1, \vec{d}_2 \in \mathcal{X}_l$ such that $\vec{d}_1 \succ_{\mathcal{N}} \vec{d}_2$, we let $(0_{l-1}, \vec{d}_1) \succ_{O^{l-1}} (0_{l-1}, \vec{d}_2)$.
- For any $\vec{d} \in \mathcal{X}_l$, we let $(0_{l-1}, \vec{d}) \succ_{O^{l-1}} (1_{l-1}, \vec{d})$.
- If $\mathcal{N} \in \{\mathcal{N}_1^l, \ldots, \mathcal{N}_{p-l+1}^l, \mathcal{N}_A^l\}$, we let $1_{l-1} 1_l \cdots 1_p \succ_{O^{l-1}} 0_{l-1} 1_l \cdots 1_{p-2} 0_{p-1} 0_p$.

We are now ready to define P_{l-1}. Any $V \in P_{l-1}$ has a counterpart in P_l. For example, the counterpart of $\hat{a}_1^{l-1}$ is $\hat{a}_1^l$. For any $V \in P_{l-1}$, V is defined to be the extension of the composition of V's counterpart in P_l and $\mathcal{N}$ (where $\mathcal{N} \in \{\mathcal{N}_1^l, \ldots, \mathcal{N}_{p-l+1}^l, \mathcal{N}_A^l, \mathcal{N}_B^l\}$), in which $1_{l-1} \cdots 1_p$ is ranked as low as possible. Next we specify which $\mathcal{N}$ that each $V \in P_{l-1}$ corresponds to.

- For any $1 \leq j \leq p$, $\hat{a}_j^{l-1}$ corresponds to $\mathcal{N}_A^l$.
- For any $1 \leq j \leq p$, $a_{(p-l)p+j}^{l-1}$ corresponds to $\mathcal{N}_j^l$.
- For any $j \leq (p-l)p$ or $(p-l+1)p+1 \leq j \leq p^2$, a_j^{l-1} corresponds to $\mathcal{N}_A^l$.
- $\hat{b}_{p-l+2}^{l-1}$ corresponds to $\mathcal{N}_A^l$.
- For any $V_B \in (B_{l-1} \cup \hat{B}_{l-1} \cup \{c\}) \setminus \{\hat{b}_{p-l+2}^{l-1}\}$, V_B corresponds to $\mathcal{N}_B^l$.

It follows that P_{l-1} is $\mathcal{O}_{l-1}$-legal, and by Lemma 6, we have the following calculation.

- For any $1 \leq j \leq p$, $|Down_{\hat{a}_j^{l-1}}(1_{l-1} \cdots 1_p)| = |Down_{\hat{a}_j^l}(1_l \cdots 1_p)| + 1$.
- For any $1 \leq j \leq p$, $|Down_{a_{(p-l)p+j}^{l-1}}(1_{l-1} \cdots 1_p)| = |Down_{a_{(p-l)p+j}^l}(1_l \cdots 1_p)| + 3$.

- For any $j \le (p-l)p$ or $(p-l+1)p+1 \le j \le p^2$, $|Down_{a_j^{l-1}}(1_{l-1}\cdots 1_p)| = |Down_{a_j^l}(1_l \cdots 1_p)| + 1$.
- $|Down_{\hat{b}^{l-1}_{p-l+2}}(1_{l-1}\cdots 1_p)| = |Down_{\hat{b}^l_{p-l+2}}(1_l \cdots 1_p)| + 1$.
- For any $V_B \in (B_{l-1} \cup \hat{B}_{l-1} \cup \{c\}) \setminus \{\hat{b}_{p-l+2}\}$, $|Down_{V_B}(1_{l-1}\cdots 1_p)| = |Down_{V_B^l}(1_l \cdots 1_p)|$, where V_B^l is the counterpart of V_B in P_l.

We next prove that $SSP_{\mathcal{O}_{l-1}}(P_{l-1}) = 1_{l-1}\cdots l_p$. We note that $P_{l-1}|_{\mathbf{x}_{l-1}=1_{l-1}} = P_l$. Therefore, if in the first step 1_{l-1} is chosen, then the winner is $1_{l-1}\cdots 1_p$. We also note that $P_{l-1}|_{\mathbf{x}_{l-1}=0_{l-1}}$ is separable (and the CP-nets are $\mathcal{N}_1^l, \ldots, \mathcal{N}_{p-l+1}^l$, p^2+p copies of $\mathcal{N}_A^l$ and p^2+p copies of $\mathcal{N}_B^l$). Therefore, if in the first step 0_{l-1} is chosen, then the winner is $0_{l-1}1_l\cdots 1_{p-2}1_{p-1}1_p$. Because exactly p^2+p-1 votes in P_{l-1} prefer $0_{l-1}1_l\cdots 1_{p-2}1_{p-1}1_p$ to $1_{l-1}\cdots 1_p$ (those votes corresponds to $\mathcal{N}_B^l$ in the construction), we have that 1_{l-1} is the winner in the first step. Therefore, $SSP_{\mathcal{O}_{l-1}}(P_{l-1}) = 1_{l-1}\cdots l_p$. It is also easy to verify that P_{l-1} satisfies all conditions in the claim.

Case 2: $p-l+1$ is odd. The construction is similar as in the even case. The only difference is that we switch the role of A_l and B_l (also $\hat{A}_l$ and $\hat{B}_l$). □

The theorem follows from Claim 2 by letting $l=1$, and it is easy to check that in P_1 in Claim 2 ($l=1$), no more than $4p^2$ alternatives has been ranked lower than $SSP_{\mathcal{O}}(P_1)$ in any vote, which means that $SSP_{\mathcal{O}}(P_1)$ is Pareto-dominated by at least $2^p - 4p^2$ alternatives.

(End of proof for Theorem 6.) □

Proof of Proposition 5: Let $P = (V_1, \ldots, V_n)$ be an $\mathcal{O}$-legal profile. Without loss of generality $Seq_{\mathcal{O}}(maj, \ldots, maj)(P) = (1_1, \ldots, 1_p)$.

First, we prove the lower bound. Because $1_1\cdots 1_p$ is the winner, for any $i \le p$, at least half of the voters prefer 1_i to 0_i, given that $\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}$ all take value 1. Therefore, by a simple counting argument as in the proof of Theorem 4, there exist $j \le n$ and a set of issues $\mathcal{I}' \subseteq \mathcal{I}$ that satisfy the following two conditions. (1) $|\mathcal{I}'| \ge p/2$, and (2) for any $\mathbf{x}_i \in \mathcal{I}'$, $1_i \succ_{V_j|\mathbf{x}_i:1_1\cdots 1_{i-1}} 0_i$, that is, voter j's preference over $\mathbf{x}_i$ is $1_i \succ 0_i$, given that $\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}$ all take value 1. For any $\vec{d} = (d_1, \ldots, d_p) \in \mathcal{X}$ such that $\vec{d}$ only takes value 1 for issues outside $\mathcal{I}'$ (and $\vec{d}$ takes value 0 for at least one issue in $\mathcal{I}'$), we next prove that $(1_1, \ldots, 1_p) \succ_{V_j} \vec{d}$. Let $\vec{d}$ be such an alternative, and $\mathbf{x}_{i_1}, \ldots, \mathbf{x}_{i_k}$ be the issues for which $\vec{d}$ takes value 0 (with $k \ge 1$, $\{\mathbf{x}_{i_1}, \ldots, \mathbf{x}_{i_k}\} \subseteq \mathcal{I}'$, and $i_1 < i_2 < \cdots < i_k$); that is, for any $l \le k$, we have $d_{i_l} = 0_l$, and for any $\mathbf{x}_i \in \mathcal{I} \setminus \{\mathbf{x}_{i_1}, \ldots, \mathbf{x}_{i_k}\}$, we have $d_i = 1_i$. We recall that for any $\mathbf{x}_i \in \mathcal{I}'$, $1_i \succ_{V_j|\mathbf{x}_i:1_1\cdots 1_{i-1}} 0_i$. Therefore, for any $l \le k$, we have the following preference relationship.

$$
\begin{aligned}
&(1_1, \ldots, 1_{i_l-1}, 1_{i_l}, d_{i_l+1}, \ldots, d_p)\\
\succ_{V_j}&(1_1, \ldots, 1_{i_l-1}, 0_{i_l}, d_{i_l+1}, \ldots, d_p)\\
=&(1_1, \ldots, 1_{i_l-1}, d_{i_l}, d_{i_l+1}, \ldots, d_p)
\end{aligned}
$$

We obtain the following preference relationship by chaining the above preference relationships.

$$
\begin{aligned}
&(1_1, \ldots, 1_p) \\
\succ_{V_j} &(1_1, \ldots, 1_{i_k-1}, 0_{i_k}, 1_{i_k+1}, \ldots, 1_p) \\
&= (1_1, \ldots, 1_{i_k-1}, d_{i_k}, d_{i_k+1}, \ldots, d_p) \\
\succ_{V_j} &(1_1, \ldots, 1_{i_{k-1}-1}, 0_{i_{k-1}}, 1_{i_{k-1}+1}, \ldots, 1_{i_k-1}, d_{i_k}, \ldots, d_p) \\
&= (1_1, \ldots, 1_{i_{k-1}-1}, d_{i_{k-1}}, d_{i_{k-1}+1}, \ldots, d_p) \\
&\vdots \\
\succ_{V_j} &(1_1, \ldots, 1_{i_1-1}, 0_{i_1}, 1_{i_1+1}, \ldots, 1_{i_2-1}, d_{i_2}, \ldots, d_p) \\
&= (d_1, \ldots, d_p) = \vec{d}
\end{aligned}
$$

Because $|\mathcal{I}'| \geq \lceil p/2 \rceil$, the number of such $\vec{d}$'s is at least $2^{\lceil p/2 \rceil} - 1$. It follows that the minimax satisfaction index is at least $2^{\lceil p/2 \rceil}$.

The upper bound follows directly from Theorem 4 and Proposition 4 (when the profile is separable, the SSP winner is the same as the Seq winner, and any separable profile must be $\mathcal{O}$-legal for any $\mathcal{O}$). □

Proof of Proposition 6:

• *Dictatorship*. The dictator always gets her most preferred alternative, and is hence maximally satisfied.

• *k-approval*. Let P be an arbitrary profile in which c_1 is ranked in the $(m+1-k)$th position from the bottom. It follows that the total score of c_1 is n. Therefore, $App_k(P) = c_1$ (we remember that ties are broken in favor of c_1).

Let P be a profile such that $App_k(P)$ is ranked within $m - k$ positions from the bottom. It follows that the total score of $App_k(P)$ is 0, and there exists $c \in \mathcal{X}$ such that the total score of c is positive. This contradicts the assumption that $App_k(P)$ is the winner.

• *Plurality with runoff*. We prove that the winner must be ranked in the top position in at least one vote. Suppose for the sake of contradiction, there exists a profile P such that $Pluo(P)$ is ranked in the top position in none of the votes in P. Because $Pluo(P)$ enters the second round, it must be the case that all the other alternatives are never ranked in the top position, with exactly one exception, denoted by c. However, c beats $Pluo(P)$ in the second round, which is a contradiction.

• *STV*. Suppose for the sake of contradiction, there exists a profile P such that $c = STV(P)$ and c is not ranked in the top position in any vote. Then, in any round, any alternative that is ranked in the first position in some vote will not be eliminated (because c is ranked in the top position in no vote). It follows that c is not the winner, which contradicts the assumption.

• *Bucklin*. Suppose for the sake of contradiction, there exists a profile P such that $c = Bucklin(P)$ and c is ranked within $\lfloor m/2 \rfloor$ positions from the bottom. Then, in each vote, there are $\lceil m/2 \rceil$ alternatives ranked in top $\lceil m/2 \rceil$ positions, which means that there exists an alternative that is ranked within top $\lceil m/2 \rceil$ positions in at least $\lceil m/2 \rceil \times n/(m-1) > n/2$ votes. It follows that the Bucklin score of that alternative is

no more than $\lceil m/2 \rceil$. We note that the Bucklin score of c is $\lceil m/2 \rceil$+1. This contradicts the assumption that c is the winner under Bucklin.
(End of proof for Proposition 6.) $\square$

Proof of Proposition 7: For any $n \in \mathbb{N}$ such that n is even, we let P be the profile in which $n/2$ votes are $c_2 \succ c_3 \succ \cdots \succ c_{\lfloor m/2 \rfloor} \succ c_1 \succ c_{\lfloor m/2+1 \rfloor} \cdots \succ c_m$, and the other $n/2$ votes are in the reversed order, that is, they are $c_m \succ c_{m-1} \succ \cdots \succ c_{\lfloor m/2+1 \rfloor} \succ c_1 \succ_{\lfloor m/2 \rfloor} \succ \cdots \succ c_2$. If m is odd, then the total score of any alternative is $n(m-1)/2$, thus c_1 is the winner. If m is even, then the total score of c_1 is $nm/2$, and the total score of any other alternative is $n(m-1)/2$, thus c_1 is the winner. It follows that $MSI_{Borda}(m,n) \leq \lfloor m/2+1 \rfloor$. We next show that $MSI_{Borda}(m,n) \geq \lfloor m/2+1 \rfloor$. Suppose for the sake of contradiction, there exists a profile P that is composed of n voters, such that $Borda(P)$ is ranked below the $\lfloor m/2+1 \rfloor$th position from the bottom. Then, the total score of $Borda(P)$ is at most $\lfloor m/2-1 \rfloor n$. However, the average total score of all alternatives is $(m-1)n/2$, which means that there exists an alternative whose total score is at least $(m-1)n/2 > \lfloor m/2-1 \rfloor n$. This contradicts with the assumption that $Borda(P)$ is the winner. It follows that $MSI_{Borda}(m,n) = \lfloor m/2 + 1 \rfloor$.

Similarly we can prove that for any $n \in \mathbb{N}$ such that $n \geq m$ and n is odd, $MSI_{Borda}(m,n) = \lceil m/2+1 \rceil$.
(End of proof for Proposition 7.) $\square$

Proof of Proposition 8: We first prove the proposition for the case in which $0 < \alpha \leq 1$ or n is even. Let P be an n-profile and $\text{Copeland}_\alpha(P) = c$. The sum of Copeland scores of the alternatives in $\mathcal{X} \setminus \{c\}$ is at least $\alpha(m-1)(m-2)/2$, which means that the Copeland score of c is at least $(\alpha(m-1)(m-2)/2)/(m-1) = \alpha(m-2)/2$. It follows that the number of draws and wins for c in the pairwise elections is at least $\alpha(m-2)/2$. Therefore, $MSI_{\text{Copeland}_\alpha}(m,n) \geq (\alpha(m-2)/2)/2+1 \geq \alpha m/4$.

Next, we prove the proposition for the case in which $\alpha = 0$, $n \geq 2m$, and n is even. Let P be an n-profile, defined as follows.

- For any $3 \leq i \leq n/2+2$, we let $V_{i-2} = [c_i \succ c_{i+1} \succ \cdots \succ c_{i+m-4} \succ c_1 \succ c_2 \succ c_{i+m-3}]$, where for any $j \in \mathbb{N}$, $c_j = c_{j+m-2}$.
- For any $3 \leq i \leq n/2+1$, we let $V_{n/2+i-2} = [c_2 \succ c_{i+m-3} \succ c_{i+m-4} \succ \cdots \succ c_{i+1} \succ c_i \succ c_1]$.
- $V_n = [c_{n/2+1} \succ c_{n/2} \succ \cdots \succ c_{n/2+3} \succ c_{n/2+2} \succ c_1 \succ c_2]$.

We observe that in P, c_1 beats c_2 in pairwise election; for any $3 \leq j \leq m$, c_j beats c_1 in pairwise election; for any $2 \leq j_1, j_2 \leq m$ with $j_1 \neq j_2$, c_{j_1} and c_{j_2} draw in pairwise election. It follows that the Copeland score of c_2 is 0, and the Copeland score of any other alternative is 1. Therefore $\text{Copeland}_0(P) = c_1$.
(End of proof for Proposition 8.) $\square$

Proof of Proposition 9: Let M be the cyclic permutation on $\{c_2, \ldots, c_m\}$ defined as follows. For any $2 \leq i \leq m$, $M(c_i) = c_{i+1}$, where $c_i = c_{i+m-1}$. For any $j \leq n$, we let $V_j = [M^j(c_2) \succ M^j(c_3) \succ \cdots M^j(c_{m-2}) \succ c_1 \succ M^j(c_{m-1}) \succ M^j(c_m)]$. We have that $\min\{N(c_1,c_i) : i \neq 1\} \geq 2\lfloor n/(m-1) \rfloor$, and for any $2 \leq i \leq m$, $\min\{N(c_i, c_{i'}) :$

$i' \neq i\} \leq \lceil n/(m-1) \rceil$. Because $n \geq m-1$, $2\lfloor n/(m-1) \rfloor \geq \lceil n/(m-1) \rceil$, which means that $Maximin(P) = c_1$.
(End of proof for Proposition 9.) □

Proof of Proposition 10: Suppose for the sake of contradiction, there exists a profile P such that $c = RankedPairs(P)$ and c is ranked lower than the $\lfloor \sqrt{m} \rfloor$th position from the bottom. It follows that there exists an alternative c' such that c' is ranked above c in at least $(m - \sqrt{m})n/m = (1 - 1/\sqrt{m})n$ votes. Because c is the winner, there exists a sequence of alternatives $d_1, \ldots, d_k$ such that c is ranked above d_1 in at least $(1 - 1/\sqrt{m})n$ votes, d_k is ranked above c' in at least $(1 - 1/\sqrt{m})n$ votes, and for any $i \leq k-1$, d_i is ranked above d_{i+1} in at least $(1 - 1/\sqrt{m})n$ votes. We let d_0 and d_{k+1} denote c and c', respectively. We prove the next claim by induction.

Claim 3 *For any $1 \leq i \leq k+1$, we have $d_0 \succ d_1 \succ \cdots \succ d_i$ in at least $n(1 - i/\sqrt{m})$ votes.*

Proof of Claim 3: The $i = 1$ case is trivial. Suppose that the claim holds for some $i \leq k$, we next show that the claim also holds for $i+1$. Because at most in $n/\sqrt{m}$ votes d_{i+1} is ranked above d_i, at least in $n(1 - i/\sqrt{m}) - n/\sqrt{m}$ votes we have $d_0 \succ d_1 \succ \cdots \succ d_i \succ d_{i+1}$. Therefore the claim holds for $i+1$. It follows that the claim holds for any $i \leq k+1$. □

By Claim 3, if $k + 1 < \sqrt{m} - 1$, then in at least $n(1 - (k+1)/\sqrt{m}) > n/\sqrt{m}$ votes we have $c \succ c'$, which contradicts the assumption that c' is ranked above c in at least $(1 - 1/\sqrt{m})n$ votes. Therefore, we must have that $k + 1 \geq \sqrt{m} - 1$, which means that $d_0 \succ d_1 \succ \cdots \succ d_{\lfloor \sqrt{m} \rfloor - 1}$ in at least $(1 - (\lfloor \sqrt{m} \rfloor - 1)/\sqrt{m})n \geq 1$ votes. This contradicts the assumption that c is ranked lower than the $\sqrt{m}$th position from the bottom.
(End of proof for Proposition 10.) □

Proof of Proposition 11: We first prove the following claim.

Claim 4 *Let T' be a voting tree, $\mathcal{X}'$ be the set of alternatives (leaf nodes) in T', $|\mathcal{X}'| = m'$.*

$$\vec{h}_* = (\underbrace{0, \ldots, 0}_{\lfloor n/2 \rfloor - m' + 1}, \underbrace{1, \ldots, 1}_{m'}, \underbrace{m'-1, \ldots, m'-1}_{\lceil n/2 \rceil - 1})$$

is realizable over $\mathcal{X}'$ through T'.

Proof of Claim 4: The proof is constructive. Without loss of generality, we let $\mathcal{X}' = \{c_1, \ldots, c_{m'}\}$. Let P be the profile that is defined as follows.

- For any $i \leq \lfloor n/2 \rfloor - m' + 1$, let $V_i = [c_{m'} \succ c_{m'-1} \succ \cdots \succ c_1]$.
- For any $i \leq m'$, let $V_{\lfloor n/2 \rfloor - m' + 1 + i} = [\mathcal{X}' \setminus (\{c_1, c_i\})] \succ c_1 \succ c_i$.
- For any $\lfloor n/2 \rfloor \leq i \leq n$, let $V_i = [c_1 \succ c_2 \succ \cdots \succ c_{m'}]$.

It is easy to check that $f_P(r_T(P)) = \vec{h}_*$. □

Then, following a similar construction as in the proof for Theorem 2, we can prove that if $l \geq 2$, then $MSI_{r_T}(m,n) = \lfloor l/2 + 2 \rfloor$; if $l = 1$, then $MSI_{r_T}(m,n) = 3$.
(End of proof for Proposition 11.) □