Voting with partial orders

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Outline

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Basic setting of voting

- **Candidates**: A set of candidates $\mathcal{X}$.
- **Voters**: A set of voters $\mathcal{A}$.
- **Preference**: $\mathcal{D}_{\mathcal{X}}$, the preference structure over candidate set $\mathcal{X}$ (linear order, subset, etc).
- **Profile**: $\mathcal{D}_{\mathcal{X}}^{\mathcal{A}}$, each voter picks a vote from the preference structure $\mathcal{D}_{\mathcal{X}}$.
- **Collective decision**: A mapping from $\mathcal{D}_{\mathcal{X}}^{\mathcal{A}}$ to specific range:
  - Rule: $r : \mathcal{D}_{\mathcal{X}}^{\mathcal{A}} \longrightarrow \mathcal{X}$.
  - Correspondence: $c : \mathcal{D}_{\mathcal{X}}^{\mathcal{A}} \longrightarrow 2^{\mathcal{X}} \backslash \emptyset$.

Why partial orders?

- Voters themselves might not be able/willing to give linear order
- Space/ Time consuming

Take partial orders as preferences

CP-net $\mathcal{N}$ [C. Boutilier *et al* 99]

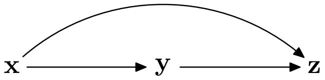
- Set of variables $\mathcal{I} = \{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$, taking values in $D_1, \ldots, D_p$

$$\text{Combinatorial domain } \mathcal{X} = D_1 \times \ldots \times D_p$$

- Directed acyclic graph $G = (\mathcal{I}, E)$
- CPT for each $\mathbf{x}_i$ indicating the conditional preference on D_i

A CP-net naturally induces a partial order over $\mathcal{X} = D_1 \times \ldots \times D_p$.

CP-net: an example



$CPT(\mathbf{x})$: $x \succ \bar{x}$

$CPT(\mathbf{y})$:
$x : y \succ \bar{y}$
$\bar{x} : \bar{y} \succ y$

$CPT(\mathbf{z})$:
$xy : z \succ \bar{z}$
$x\bar{y} : z \succ \bar{z}$
$\bar{x}y : z \succ \bar{z}$
$\bar{x}\bar{y} : \bar{z} \succ z$

The partial order $\mathcal{N}$ induces is

$$xyz \nearrow x\bar{y}z \searrow, \quad xyz \searrow xy\bar{z} \nearrow \quad x\bar{y}\bar{z} \longrightarrow \bar{x}\bar{y}\bar{z} \longrightarrow \bar{x}\bar{y}z \longrightarrow \bar{x}yz \longrightarrow \bar{x}y\bar{z}$$

Extension of partial order

A linear order V on $\mathcal{X}$ is said to

- extend a partial order P, if $P \subseteq V$. We say V is an extension (completion) of P
- extend a CP-net $\mathcal{N}$ over $\mathcal{X}$, if V extends the partial order that $\mathcal{N}$ induces
- be compatible with a linear order $\mathcal{O}$ on $\mathcal{I}$, if it extends a CP-net that is compatible with $\mathcal{O}$

Voting on combinatorial domain with preferences modeled by CP-nets [J.Lang 07]

- A set of issues $\mathcal{I} = \{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ on $\mathcal{X} = D_1 \times \ldots \times D_p$.
- A linear order $\mathcal{O}$ over $\mathcal{I}$, for example $\mathcal{O} = \mathbf{x}_1 > \ldots > \mathbf{x}_p$.
- p local voting rules $r_1, \ldots, r_p$.
- Input: A profile $P = (V_1, \ldots, V_N)$ s.t. V_j is compatible with $\mathcal{O}$.

Output: $(d_1, \ldots, d_p) \in \mathcal{X}$ through a p-step process

1. Select d_1 by r_1 from $P|_{\mathbf{x}_1}$.
2. Select d_2 by r_2 from $P|_{\mathbf{x}_2|\mathbf{x}_1=d_1}$.

   $\vdots$

p. Select d_p by r_p from $P|_{\mathbf{x}_p|\mathbf{x}_1=d_1,\ldots,\mathbf{x}_{p-1}=d_{p-1}}$.

Such a rule is defined to be the *sequential composition* of $r_1, \ldots, r_p$, denoted by $Seq(r_1, \ldots, r_p)$. It is said to be *decomposable*.
Special case: Seat-by-seat voting: No edges in CP-net, all issues are voted on separately

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Question: Is there any neutral or Pareto efficient decomposable voting rules or correspondences?

- Neutrality: the voting rule is insensitive to any permutation of candidates
- Pareto efficiency: for any two candidates c_1, c_2, if c_1 is preferred to c_2 by all voters, then c_2 cannot be the winner

Impossibility theorems by J. Benoit and L. Kornhauser 06

1. If the domain is not the product of two binary issues, then the only Pareto efficient seat-by-seat voting rule is a dictatorship.
   i.e. $\{a_1, b_1\} \times \{a_2, b_2, c_2\}$.
2. If there are three or more issues, and each local rule satisfies efficiency, then the only neutral seat-by-seat voting rule is a dictatorship.
   i.e. $\{a_1, b_1\} \times \{a_2, b_2\} \times \{a_3, b_3\}$.

Our impossibility theorems

1. If the domain is not the product of two binary issues, then the only Pareto efficient seat-by-seat voting rule or correspondence is a dictatorship or a trivial one.
2. If the domain is not the product of two binary issues, then the only neutral seat-by-seat voting rule or correspondence is a dictatorship, or an anti-dictatorship, or a trivial one.

More on neutrality and Pareto efficiency

- Sequential composition of two plurality rules on two binary issues is neutral and Pareto efficient[1]
- The theorems can be easily extended to sequential voting rules/ correspondences

[1] L.Xia, J. Lang, M. Ying 07.

Outline

Possible (necessary) winner, [Konczak and Lang, 2005]

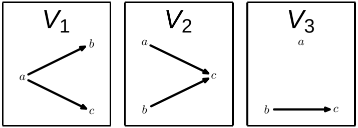
Definition

A candidate c is a *possible (necessary) winner* for a set of partial orders P, w.r.t. voting rule r, if there exists a (for any) set of linear orders P' extending P, $r(P') = c$.

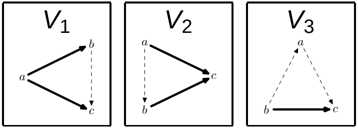
Need to distinguish between unique winner and co-winner ($c \in r(P')$).

Example: Possible (unique) winner under Plurality rule

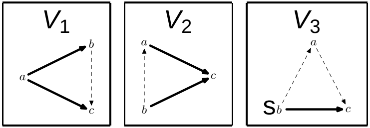
Preferences:



a is a possible (unique) winner:

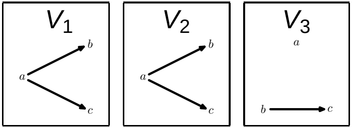


a is not a necessary winner:



Example: Necessary (unique) winner

Preferences:



Candidate *a* will always win!

Possible/necessary unique/co-winner determination

Input:

1. A voting rule r
2. A candidate c
3. A set of partial orders P.

Question: is c is a possible/necessary unique/co-winner?

The votes are unweighted.

Complexity results

Voting rule	Possible winner	Necessary winner
Scoring[1]	NP-hard	$O(nm^2)$
STV	NP-hard[2]	coNP-hard[2]
Copeland[1]	NP-hard	coNP-hard
Maximin[1]	NP-hard	$O(nm^3)$
Bucklin[1]	NP-hard	$O(nm^2)$
Ranked pairs[1]	NP-hard	coNP-hard

n: # votes

m: # candidates

- Holds for both unique/co-winner.

[1] Even when the # incomparable(unknown) pairs in each vote is small (less than 16)

[2] Even when the partial orders are modeled by CP-nets.

Summary

- Impossibility theorem about seat-by-seat voting
- Complexity of computing possible/necessary unique/co-winners given partial orders

Future work

- Possible/necessary unique/co-winners when inputs are CP-nets
- Is it usually easy to compute possible/necessary unique/co-winners