Strongly Decomposable Voting Rules on Multiattribute Domains

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Abstract

Sequential composition of voting rules, by making use of structural properties of the voters' preferences, provide computationally economical ways for making a common decision over a Cartesian product of finite local domains. A sequential composition is usually defined on a set of legal profiles following a fixed order. In this paper, we generalize this by *order-independent sequential composition* and *strong decomposability*, which are independent of the chosen order. We study to which extent some usual properties of voting rules transfer from the local rules to their order-independent sequential composition. Then, to capture the idea that a voting rule is neutral or decomposable on a slightly smaller domain, we define *nearly neutral, nearly decomposable* rules for both sequential composition and order-independent sequential composition, which leads us to defining and studying *decomposable permutations*. We prove that any sequential composition of neutral local rules and any order-independent sequential composition of neutral local rules satisfying a necessary condition are nearly neutral.

Introduction

When the set of candidates has a combinatorial structure, the space needed for storing a preference relation increases exponentially. To overcome this problem, several approaches were designed to exploit and use the independence information in a preference relation, leading to concise representations, especially, CP-nets (Boutilier *et al.* 2004). In (Lang 2007), a sequential voting process was suggested, consisting of local voting rules or correspondences, the winner being selected through multiple steps from a set of votes satisfying some independence conditions. Such admissible input profiles are referred to as *legal* profiles. A rule or correspondence is said to be decomposable, if its restriction to legal profiles is the sequential composition of local rules on respective subdomains. In (Xia, Lang, & Ying 2007) it is proved that *anonymity, homogeneity, neutrality, participation, consensus* are inherited to local rules from their sequential composition, *monotonicity* are inherited to the last local rule, and *consistency* is also inherited if the sequential composition satisfies homogeneity. On the other hand, only *anonymity, homogeneity, consistency* can be lifted from local rules to their sequential composition, while *monotonicity* can be lifted from the last local rule. An especially important property is *neutrality*. Although it has been proved in (Xia, Lang, & Ying 2007) that the sequential composition of two binary plurality rules (resp. correspondences) is neutral, some negative results arise. For example, if a local domain has more than three candidates, then the sequential composition of plurality rules (resp. correspondences) is not neutral. It has also been proved that sequential composition on any rules cannot satisfies both neutrality and the Condorcet criterion. It is still unknown whether there exists a neutral decomposable rule or correspondence other than the sequential composition of two plurality rules on binary subdomains.

In this paper, we define the sequential composition of local rules over a domain of "legal" profiles that do not require the order on which the local rules are applied to be fixed from the beginning of the process. Such a composition is said to be *order-independent*, because it is, to some extent, insensitive to the order in which the local rules are applied; the order-independent sequential composition of local rules is said to be strongly decomposable. Because strong decomposability is stronger than decomposability, not all results on decomposability can be directly carried over to strongly decomposable case. Therefore we study the relation between properties that local rules and their order-independent sequential composition satisfy respectively. For the specific case of neutrality, we first study a specific class of permutations on multiattribute domains, called *decomposable permutations*. However, since directly proving or disproving the existence of a neutral decomposable rule is hard, we slightly relax the domain of application of decomposability and neutrality, and introduce *nearly neutral* and *nearly decomposable* rules. We show that every sequential composition of neutral local correspondences is nearly neutral. These results can be extended to strong decomposability.

The paper is structured as follows. First we recall some basics on CP-nets, decomposable voting rules and properties of voting rules. Then we introduce order-independent sequential composition and strong decomposability, and address next the relation between local rules and their order-independent sequential composition. Then, we study permutations between legal profiles following different orders, which enable us to define nearly neutral and nearly decomposable rules and correspondences, and we give our main results. Because of space limit, proofs are omitted.

Notations and basic definitions

CP-nets and structured preferences

Let $\mathcal{A} = \{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ be a set of variables (or attributes), and D_i being the finite value domain of $\mathbf{x}_i$. Let $\mathcal{X} = D_1 \times \ldots \times D_p$. $\mathcal{X}$ is a combinatorial (or multiattribute) domain. A *CP-net* over $\mathcal{A}$ is composed of (a) directed acyclic graph (DAG) G over $\mathbf{x}_1, \ldots, \mathbf{x}_p$ and (b) a set of conditional linear preference orders over D_i associated to each variable $\mathbf{x}_i$, expressed by a conditional preference table $CPT(\mathbf{x}_i)$ consisting of a linear preference order $\succ^i_{\vec{u}}$ over D_i for each tuple of values $\vec{u}$ for the parents of $\mathbf{x}_i$ in G.

Given a CP-net $\mathcal{N}$, a linear preference V over $\mathcal{X}$ is said to extend $\mathcal{N}$, denoted by $V \sim \mathcal{N}$, if for any i, any $\succ^i_{\vec{u}} \in CPT(\mathbf{x}_i)$, and any $\vec{x} \in \prod_{\mathbf{x}_j \notin \{\mathbf{x}_i\} \cup Par(\mathbf{x}_i)} D_j$,

$$(x_i, \vec{u}, \vec{x}) \succ_V (y_i, \vec{u}, \vec{x}) \text{ iff } x_i \succ^i_{\vec{u}} y_i.$$

This definition captures the conditional independence of linear orders over $\mathcal{X}$. Namely, if V extends $\mathcal{N}$, then for any i, given the value of $Par(\mathbf{x}_i)$, the preference over D_i is independent of all non-descendent variables of $\mathbf{x}_i$. The set of all CP-nets on $\mathcal{X}$ is denoted by $CP(\mathcal{X})$.

Given an ordering $\mathcal{O} = \mathbf{x}_{\sigma(1)} > \ldots > \mathbf{x}_{\sigma(p)}$ of V, where σ is a permutation of $\{1, \ldots, p\}$, we say a DAG G is compatible with $\mathcal{O}$, denoted as $G \sim \mathcal{O}$, if for any $\mathbf{x}_i >_{\mathcal{O}} \mathbf{x}_j$, $\mathbf{x}_j$ is not an ancestor of $\mathbf{x}_i$ in G. A CP-net $\mathcal{N}$ is said to be compatible with $\mathcal{O}$, denoted by $\mathcal{N} \sim \mathcal{O}$, if its DAG is compatible with $\mathcal{O}$. The set of all CP-nets compatible with $\mathcal{O}$ is denoted by $CP(\mathcal{O})$.

We say a linear preference V is compatible with $\mathcal{O}$, denoted by $V \sim \mathcal{O}$, if there exists a CP-net $\mathcal{N}$ compatible with $\mathcal{O}$ such that V extends $\mathcal{N}$. Clearly, in a CP-net compatible with $\mathcal{O}$, $Par(\mathbf{x}_{\sigma(i)}) \subseteq \{\mathbf{x}_{\sigma(1)}, \ldots, \mathbf{x}_{\sigma(i-1)}\}$. Therefore, for any linear preference V compatible with $\mathcal{O}$, if the value of $\mathbf{x}_{\sigma(1)}, \ldots, \mathbf{x}_{\sigma(i-1)}$ is given, then the local preference over $D_{\sigma(i)}$ is fixed. We write $V^{\mathbf{x}_{\sigma(i)}|d_{\sigma(1)}\cdots d_{\sigma(i-1)}}$ for the conditional preference over $D_{\sigma(i)}$ given $\mathbf{x}_{\sigma(1)} = d_{\sigma(1)}, \ldots, \mathbf{x}_{\sigma(i-1)} = d_{\sigma(i-1)}$, and $P^{\mathbf{x}_{\sigma(i)}|d_{\sigma(1)}\cdots d_{\sigma(i-1)}} = \{V^{\mathbf{x}_{\sigma(i)}|d_{\sigma(1)}\cdots d_{\sigma(i-1)}} : V \in P\}$.

Decomposable voting rules and correspondences

Given $\mathcal{X}$ a finite set of candidates, a *profile* of N votes over $\mathcal{X}$ is a sequence of N linear orders over $\mathcal{X}$, denoted by $P = (V_1, \ldots, V_N)$. The set of all profiles over $\mathcal{X}$ is denoted by $P_{\mathcal{X}}$. A voting rule r over $\mathcal{X}$ is a function that maps each profile P to $r(P) \in \mathcal{X}$, where $r(P)$ is referred to as the winner of P. A voting correspondence c over $\mathcal{X}$ selects a nonempty set of winners from a profile, thus is a mapping $P_{\mathcal{X}} \to 2^{\mathcal{X}} \backslash \{\emptyset\}$.

Given a multi-attribute domain $\mathcal{X} = D_1 \times \ldots \times D_p$, a decomposable voting rule (Lang 2007) is a voting rule defined over all profiles that are compatible with a given order $\mathcal{O}$. We refer to such profiles $\mathcal{O}$-legal profiles.

Definition 1 (**$\mathcal{O}$-legal**) *A vote V on $\mathcal{X} = D_1 \times \ldots \times D_p$ is $\mathcal{O}$-legal if V is compatible with $\mathcal{O}$. The set of all $\mathcal{O}$-legal votes is denoted $Legal(\mathcal{O})$.*

A profile P is $\mathcal{O}$-legal if all of its votes are $\mathcal{O}$-legal. We write $Legal_\infty(\mathcal{O}) = \bigcup_{i=1}^\infty Legal(\mathcal{O})^i$ to represent the set of all $\mathcal{O}$-legal profiles. We also write $Legal_i(\mathcal{X})$ to represent the set of all legal profiles of i voters, and $Legal(\mathcal{X})$ to represent the set of all legal profiles. By definition, $Legal_i(\mathcal{X}) = \bigcup_{\mathcal{O}} Legal(\mathcal{O})^i$ and $Legal(\mathcal{X}) = \bigcup_{i=1}^\infty Legal_i(\mathcal{X})$.

Example 1 *Let $p = 2$ and $D_i = \{0_i, 1_i\}$, $i = 1, 2$. Consider the following votes:*

V_1 $1_1 1_2 \succ 1_1 0_2 \succ 0_1 1_2 \succ 0_1 0_2$
V_2 $1_1 1_2 \succ 1_1 0_2 \succ 0_1 0_2 \succ 0_1 1_2$
V_3 $1_1 1_2 \succ 0_1 1_2 \succ 0_1 0_2 \succ 1_1 0_2$
V_4 $1_1 1_2 \succ 0_1 0_2 \succ 1_1 0_2 \succ 0_1 1_2$

V_1 extends the CP-net $1_1 \succ 0_1, 0_1 : 1_2 \succ 0_2, 1_1 : 1_2 \succ 0_2$, thus it is in $Legal(\mathbf{x}_1 > \mathbf{x}_2)$, and therefore in $Legal(\mathcal{X})$. It is also in $Legal(\mathbf{x}_2 > \mathbf{x}_1)$, because it extends the CP-net $1_2 \succ 0_2, 0_2 : 1_1 \succ 0_1, 1_2 : 1_1 \succ 0_1$. V_2 is in $Legal(\mathbf{x}_1 > \mathbf{x}_2)$, and thus in $Legal(\mathcal{X})$, but not in $Legal(\mathbf{x}_2 > \mathbf{x}_1)$. V_3 is in $Legal(\mathbf{x}_2 > \mathbf{x}_1)$, and thus in $Legal(\mathcal{X})$ but not in $Legal(\mathbf{x}_1 > \mathbf{x}_2)$. V_4 is not in $Legal(\mathbf{x}_1 > \mathbf{x}_2)$ nor in $Legal(\mathbf{x}_2 > \mathbf{x}_1)$, thus it is not in $Legal(\mathcal{X})$.

The 2-voter profile $\{V_1, V_2\}$ is in $Legal(\mathbf{x}_1 > \mathbf{x}_2)^2$, therefore it is legal (i.e., it is in $Legal_2(\mathcal{X})$). The 2-voter profile $\{V_2, V_3\}$ is not legal, although both V_2 and V_3 are legal, simply because there is no common ordering $\mathcal{O}$ such that V_2 and V_3 are both $\mathcal{O}$-legal.

Then we recall the definition of $\mathcal{O}$-sequential composition of voting rules (Lang 2007). Given an order $\mathcal{O} = \mathbf{x}_{\sigma(1)} > \ldots > \mathbf{x}_{\sigma(p)}$ and a set of local rules $\{r_1, \ldots, r_p\}$, with r_i over D_i, their $\mathcal{O}$-sequential composition $Seq(r_{\sigma(1)}, \ldots, r_{\sigma(p)})$ is defined to be a p-step voting rule over all $\mathcal{O}$-legal profiles. Given an $\mathcal{O}$-legal profile P, in the first step $r_{\sigma(1)}$ selects $d_{\sigma(1)}$ from $P^{\mathbf{x}_{\sigma(1)}}$, and after $d_{\sigma(1)}, \ldots, d_{\sigma(i-1)}$ have been selected, $d_{\sigma(i)}$ is selected by $r_{\sigma(i)}$ from $P^{\mathbf{x}_{\sigma(i)}|d_{\sigma(1)}\cdots d_{\sigma(i-1)}}$. After p steps, $(d_{\sigma(1)}, \ldots, d_{\sigma(p)})$ is chosen to be the winner. The following is the formal definition.

Definition 2 *For any local rules $\{r_1, \ldots, r_p\}$ and an order $\mathcal{O}$, define their $\mathcal{O}$-sequential composition $Seq(r_{\sigma(1)}, \ldots, r_{\sigma(p)})$ be a rule over $Legal_\infty(\mathcal{O})$ s.t. for any $\mathcal{O}$-legal profile P, $Seq(r_{\sigma(1)}, \ldots, r_{\sigma(p)})(P) = (d_{\sigma(1)}, \ldots, d_{\sigma(p)})$ iff for all $i \leq p$,*

$$d_{\sigma(i)} = r_{\sigma(i)}(P^{\mathbf{x}_{\sigma(i)}|d_{\sigma(1)}\cdots d_{\sigma(i-1)}}).$$

The $\mathcal{O}$-sequential composition of correspondences is defined similarly. The difference is, at each step, c_i selects multiple winners.

Definition 3 *For any local correspondences $c_1, \ldots, c_p$, define their $\mathcal{O}$-sequential composition $Seq(c_{\sigma(1)}, \ldots, c_{\sigma(p)})$ as a correspondence over $Legal_\infty(\mathcal{O})$ s.t. for any $\mathcal{O}$-legal profile P, $(d_{\sigma(1)}, \ldots, d_{\sigma(p)}) \in Seq(c_{\sigma(1)}, \ldots, c_{\sigma(p)})(P)$ iff for all $i \leq p$,*

$$d_{\sigma(i)} \in c_{\sigma(i)}(P^{\mathbf{x}_{\sigma(i)}|d_{\sigma(1)}\cdots d_{\sigma(i-1)}}).$$

Now we recall the definition of *decomposable* voting rules. A voting rule is decomposable iff it can be written as a sequential composition of multiple local rules on $Legal_\infty(\mathcal{O})$ for some order $\mathcal{O}$.

Definition 4 *A voting rule r on $\mathcal{X} = D_1 \times \ldots \times D_p$ is* decomposable *iff there exist p voting rules $r_1, \ldots, r_p$ on $D_1, \ldots, D_p$ and an order $\mathcal{O}$ on $\mathcal{X}$ such that for any $\mathcal{O}$-legal profile P, we have $Seq(r_{\sigma(1)}, \ldots, r_{\sigma(p)})(P) = r(P)$. The definition is similar for correspondences.*

Properties of voting rules

In this section we briefly recall some well-known criteria for voting rules. A voting rule r satisfies

- *anonymity*, if the output of the rule is insensitive to a permutation of voters;
- *homogeneity*, if for any vote V and any $n \in \mathbb{N}$, $r(V) = r(nV)$;
- *neutrality*, if for any profile P and any permutation M on candidates, $r(M(P)) = M(r(P))$;
- *monotonicity*, if for any profile $P = (V_1, \ldots, V_N)$ and another profile $P' = (V'_1, \ldots, V'_N)$ s.t. each V'_i is obtained from V_i by raising only $r(P)$, we have $r(P') = r(P)$;
- *consistency, also known as reinforcement*, if for any two disjoint profiles P_1, P_2 s.t. $r(P_1) = r(P_2)$, then $r(P_1 \cup P_2) = r(P_1) = r(P_2)$;
- *participation*, if for any profile P and any vote V, $r(P \cup \{V\}) \succcurlyeq_V r(P)$;
- *consensus* if for any profile $P = (V_1, \ldots, V_N)$, there is no candidate c s.t. $c \succ_{V_i} r(P)$ for all $i \leq N$;
- *Condorcet criterion*, if whenever there is a Condorcet winner in a voting profile P, then $r(P)$ must be the Condorcet winner.

Order-independent sequential composition

The sequential composition of rules as defined in the previous Section assumes that the order $\mathcal{O}$ according to which the voters have to report their conditional preferences on variable domains is fixed from the beginning. This is a strong restriction, as in many contexts, this order is not known from the beginning of the process. Therefore we consider the following notion, that does not need the order to be fixed. In the sequel we always write $\mathcal{O} = \mathbf{x}_{\sigma(1)} > \ldots > \mathbf{x}_{\sigma(p)}$,$\mathcal{O}' = \mathbf{x}_{\gamma(1)} > \ldots > \mathbf{x}_{\gamma(p)}$.

Definition 5 (order-independent sequential composition) *Given a set of voting rules $\{r_1, \ldots, r_p\}$ over $D_1, \ldots, D_p$, their* order-independent sequential composition *is defined as mapping from $Legal(\mathcal{X})$ to $\mathcal{X}$ such that for any order $\mathcal{O}$ and $P \in Legal(\mathcal{O})$,*

$$Seq^{OI}(r_1, \ldots, r_p)(P) = Seq(r_{\sigma(1)}, \ldots, r_{\sigma(p)})(P).$$

$Seq^{OI}(r_1, \ldots, r_p)$ is well defined, because it has been proved in (Lang 2007) (Observation 3) that for any $P \in Legal(\mathcal{X})$, if $P \sim \mathcal{O}$ and $P \sim \mathcal{O}'$ then $Seq(r_{\sigma(1)}, \ldots, r_{\sigma(p)})(P) = Seq(r_{\gamma(1)}, \ldots, r_{\gamma(p)})(P)$.

"Order-independent" means that the ordering of variables $\mathcal{O}$ is not fixed from the beginning, and once the order is given, then order-independent sequential composition is indeed the sequential composition of the order. The difference between order-independent and fixed-order sequential compositions of voting rules is in their *applicability domains*: while $Seq(r_1, \ldots, r_p)$ is defined only on $Legal(\mathbf{x}_1 > \ldots > \mathbf{x}_p)$, $Seq^{OI}(r_1, \ldots, r_p)$ is defined on the set $Legal(\mathcal{X})$ of all legal profiles.

We now strengthen the notion of decomposability so that it applies on order-independent sequential composition. A voting rule is strongly decomposable if its restrictions on $Legal(\mathcal{X})$ is the order-independent sequential composition of some local rules.

Definition 6 (Strong decomposability) *A voting rule r on $\mathcal{X} = D_1 \times \ldots \times D_p$ is strongly decomposable iff there exist voting rules $r_1, \ldots, r_p$ on $D_1, \ldots, D_p$ such that for any legal profile P, we have $Seq^{OI}(r_{\sigma(1)}, \ldots, r_{\sigma(p)})(P) = r(P)$. The definition for correspondences is similar.*

From the definition of strong decomposability we immediately know that if r is strongly decomposable, then it is also decomposable. For each of the properties of voting rules listed above, we now consider the logical relationship between the satisfaction of the property for each of the local rules and the satisfaction of the property for their order-independent sequential composition. The following result states that for most of these properties, if at least one r_i does not satisfy it then the sequential composition does not either (see (Xia, Lang, & Ying 2007) for similar results for fixed-order composition).

Theorem 1 *Let* $\texttt{Prop} \in$ *{anonymity, homogeneity , neutrality, monotonicity, consistency, participation, consensus}. If $Seq^{OI}(r_1, \ldots, r_p)$ satisfies* Prop *then for any $1 \leq i \leq p$, r_i also satisfies* Prop.

We then consider the implication in the reverse direction.

Theorem 2 *Let* $\texttt{Prop} \in$ *{homogeneity, monotonicity, consistency}. If for all $1 \leq i \leq p$, r_i satisfies* Prop *then $Seq^{OI}(r_1, \ldots, r_p)$ also satisfies* Prop.

We now focus on neutrality. We start by the specific case of two binary variables. It is already known(Xia, Lang, & Ying 2007) that the composition of two plurality correspondences on binary domains is neutral. This extends to order-independent composition:

Theorem 3 *Let c_1 (resp. c_2) be the plurality correspondence on $\{0_1, 1_1\}$ (resp. on $\{0_2, 1_2\}$). Then $Seq^{OI}(c_1, c_2)$ is a neutral correspondence.*

By theorem 1, the neutrality of order-independent sequential composition induces the neutrality of each r_i. Now we present another necessary condition for $Seq^{OI}(c_1, \ldots, c_p)$ to be neutral.

Theorem 4 *If $Seq^{OI}(c_1, \ldots, c_p)$ is neutral, then*

$$(|D_i| = |D_j|) \Rightarrow (c_i = c_j).$$

Here $c_i = c_j$ means that c_i and c_j behave the same on respective domain: for any bijection $f_{i,j} : D_i \to D_j$ and any profile P_i on D_i, $f_{i,j}(c_i(P_i)) = c_j(f_{i,j}(P_i))$. This notation is meaningful because c_i and c_j are neutral and $|D_i| = |D_j|$.

Decomposable permutations

Analyzing the neutrality of (strongly) decomposable voting rules is difficult, mainly because of the domain restriction of such rules: the problem relies in the fact that the effect of a transformation on a legal profile may not be legal. Therefore, we study the permutations that transform a legal profile into another legal one. Since the outcome of a sequential rule is determined by the CP-nets the votes are consistent with, we focus on pairs of the CP-nets $(\mathcal{N}_1, \mathcal{N}_2)$, $\mathcal{N}_1 \sim \mathcal{O}, \mathcal{N}_2 \sim \mathcal{O}'$ s.t. there exists a permutation M and a vote $V_1 \sim \mathcal{N}_1$ and $M(V_1) \sim \mathcal{N}_2$. We first study the case $\mathcal{O} = \mathcal{O}'$, and then extend the results to $\mathcal{O} \neq \mathcal{O}'$.

Order preserving permutations

We first define a class of permutations composed of multiple steps (similarly to sequential voting rules). For any set X, let $S(X)$ be the set of all permutations on X. To better present the properties of decomposable permutations, we give the following definition so as to describe a permutation that can transform a linear preference extending a given CP-net to a linear preference that is compatible with $\mathcal{O}$.

Definition 7 ($(\mathcal{N}, \mathcal{O})$-legal) *Let $\mathcal{N}$ be a CP-net over $\mathcal{X}$. A permutation $M \in S(\mathcal{X})$ is $(\mathcal{N}, \mathcal{O})$-legal if there exists a vote V extending $\mathcal{N}$ and $M(V)$ is $\mathcal{O}$-legal.*

We now define $\mathcal{O}$-decomposable permutations. A $\mathcal{O}$-decomposable M is composed of a set of conditional permutations $\{M_i^{\vec{d}_i} \in S(D_{\sigma(i)}) : i \leq p, \vec{d}_i \in D_{\sigma(1)} \times \ldots \times D_{\sigma(i-1)}\}$, and transform $\vec{d} = (d_{\sigma(1)}, \ldots, d_{\sigma(p)})$ in p steps. In the first step, $d_{\sigma(1)}$ is transformed to $M_1^{\emptyset}(d_{\sigma(1)})$, which is the $D_{\sigma(1)}$-component of $M(\vec{d})$. After the first $i-1$ steps are complete, $d_{\sigma(i)}$ is transformed by $M_i^{d_{\sigma(1)}, \ldots, d_{\sigma(i-1)}}$. The process ends after p steps.

Definition 8 ($\mathcal{O}$-decomposable permutation) *A permutation $M \in S(\mathcal{X})$ is $\mathcal{O}$-decomposable for $\mathcal{O} = \mathbf{x}_{\sigma(1)} > \ldots > \mathbf{x}_{\sigma(p)}$, if for each $1 \leq i \leq p$ and each $\vec{d}_i \in D_{\sigma(1)} \times \ldots \times D_{\sigma(i-1)}$, there exists a permutation $M_i^{\vec{d}_i}$ on $D_{\sigma(i)}$ s.t.*

$$M(d_{\sigma(1)}, \ldots, d_{\sigma(p)}) = (M_1^{\emptyset}(d_{\sigma(1)}), \ldots, M_p^{(d_{\sigma(1)}, \ldots, d_{\sigma(p-1)})}(d_{\sigma(p)})).$$

The set of all $\mathcal{O}$-decomposable permutation is denoted by $DP(\mathcal{O})$.

Example 2 *Let $p = 2$, $D_1 = \{0_1, 1_1\}$, $D_2 = \{0_2, 1_2, 2_2\}$, and $\mathcal{O} = \mathbf{x}_1 > \mathbf{x}_2$. Consider the permutation M: $0_1 0_2 \mapsto 1_1 1_2$; $0_1 1_2 \mapsto 1_1 2_2$; $0_1 2_2 \mapsto 1_1 0_2$; $1_1 0_2 \mapsto 0_1 0_2$; $1_1 1_2 \mapsto 0_1 2_2$; $1_1 2_2 \mapsto 0_1 1_2$. M is $\mathcal{O}$-decomposable. Its local conditional permutations are: $M_1(0_1) = 1_1$; $M_1(1_1) = 0_1$; $M_2^{\mathbf{x}_1 = 0_1}(0_2) = 1_2$; $M_2^{\mathbf{x}_1 = 0_1}(1_2) = 2_2$; $M_2^{\mathbf{x}_1 = 0_1}(2_2) = 0_2$; $M_2^{\mathbf{x}_1 = 1_1}(0_2) = 0_2$; $M_2^{\mathbf{x}_1 = 1_1}(1_2) = 2_2$; $M_2^{\mathbf{x}_1 = 1_1}(2_2) = 1_2$.*

The following question naturally arises: for any $M \in DP(\mathcal{O})$, if V extends $\mathcal{N}$, then what is the CP-net that $M(V)$ extends? The answer is a CP-net obtained by $\mathcal{N}$ after a special permutation closely related to M. To define this permutation, we write $Ind(M, i)$ to represent the temporary winner after first i steps of a decomposable permutation M.

Definition 9 *For any $M \in DP(\mathcal{O})$ and any $i \leq p$, define an induced permutation $Ind(M, i)$ on $\prod_{j=1}^{i} D_{\sigma(j)}$ s.t. for any $d_{\sigma(j)} \in D_{\sigma(j)}, j \leq i$,*

$$Ind(M, i)(d_{\sigma(1)}, \ldots, d_{\sigma(i)}) = (M_1^{\emptyset}(d_{\sigma(1)}), M_2^{d_{\sigma(1)}}(d_{\sigma(2)}), \ldots, M_i^{d_{\sigma(1)}, \ldots, d_{\sigma(i-1)}}(d_{\sigma(i)})).$$

Then we define the permutation on CP-nets induced by M.

Definition 10 *Define a mapping $f_{\mathcal{O}} : DP(\mathcal{O}) \to S(CP(\mathcal{O}))$ such that for any $\mathcal{O}$-decomposable permutation M and any $\mathcal{N} \in CP(\mathcal{O})$, if $x_{\sigma(1)}, \ldots, x_{\sigma(i)} : y_{\sigma(i+1)} \succ_{\mathcal{N}} z_{\sigma(i+1)}$, then*

$$Ind(M, i)(x_{\sigma(1)}, \ldots, x_{\sigma(i)}) : M_{i+1}^{x_{\sigma(1)}, \ldots, x_{\sigma(i)}}(y_{\sigma(i+1)}) \succ_{f_{\mathcal{O}}(M)(\mathcal{N})} M_{i+1}^{x_{\sigma(1)}, \ldots, x_{\sigma(i)}}(z_{\sigma(i+1)})$$

Example 3 *Take M as in Example 2. Consider the CP-net $\mathcal{N}$: $0_1 \succ 1_1$; $0_1 : 0_2 \succ 1_2 \succ 2_2$; $1_1 : 1_2 \succ 2_2 \succ 0_2$. Then $f_{\mathcal{O}}(M)(\mathcal{N})$ is the following CP-net: $1_1 \succ 0_1$; $0_1 : 2_2 \succ 1_2 \succ 0_2$; $1_1 : 1_2 \succ 2_2 \succ 0_2$.*

The next three theorems shed some light on the "legal pairs" $(P, M(P))$. The first and second concern the case where $M \in DP(\mathcal{O})$, and the third concerns the case where $M \notin DP(\mathcal{O})$.

The first theorem gives a characterization of the CP-net associated with a vote obtained after applying a decomposable permutation. It says that for any $\mathcal{O}$-decomposable permutation M, if V is compatible with a CP-net $\mathcal{N}$ compatible with $\mathcal{O}$, and $M(V)$ is also $\mathcal{O}$-legal, then $M(V)$ must extend $f_{\mathcal{O}}(M)(\mathcal{N})$.

Theorem 5 *For any $M \in DP(\mathcal{O})$ and any CP-net $\mathcal{N} \sim \mathcal{O}$, if a vote V extends $\mathcal{N}$ and $M(V) \sim \mathcal{O}$, then $M(V) \sim f_{\mathcal{O}}(M)(\mathcal{N})$.*

Thus, in Example 3, if we take $V = 0_1 0_2 \succ 0_1 1_2 \succ 1_1 1_2 \succ 0_1 2_2 \succ 1_1 2_2 \succ 1_1 0_2$. $M(V) = 1_1 1_2 \succ 1_1 2_2 \succ 0_1 2_2 \succ 1_1 0_2 \succ 0_1 1_2 \succ 0_1 0_2$. We have that $V \sim \mathcal{O}$ and $M(V) \sim \mathcal{O}$, therefore $M(V)$ extends the CP-net $f_{\mathcal{O}}(M)(\mathcal{N})$.

The next theorem focuses on decomposability. It says that the composition of neutral local correspondences is insensitive to permutations in $DP(\mathcal{O})$. The same theorem holds for decomposable rules.

Theorem 6 *Let $c_1, \ldots, c_p$ be neutral correspondences on $D_1, \ldots, D_p$, respectively. For any $\mathcal{O}$-legal profile P and any $M \in DP(\mathcal{O})$, if $M(P)$ is $\mathcal{O}$-legal, then $M(Seq(c_1, \ldots, c_p)(P)) = Seq(c_1, \ldots, c_p)(M(P))$.*

Notice that the precondition in this theorem requires both P and $M(P)$ are $\mathcal{O}$-legal. This does not mean for any $M \in DP(\mathcal{O})$, $M(P)$ is $\mathcal{O}$-legal for all $\mathcal{O}$-legal profiles P. In fact, $M(P)$ is not necessarily legal, for example, consider $V_1 = 1_1 1_2 \succ 0_1 1_2 \succ 1_1 0_2 \succ 0_1 0_2 \in Legal(\mathbf{x}_1 > \mathbf{x}_2)$, $M \in DP(\mathbf{x}_1 > \mathbf{x}_2)$ s.t. it only exchanges $1_1 0_2$ and $1_1 1_2$. Then $M(V_1) = 1_1 0_2 \succ 0_1 1_2 \succ 1_1 1_2 \succ 0_1 0_2 \notin Legal(\mathcal{X})$.

The last theorem says that if $M \notin DP(\mathcal{O})$, then there exists a CP-net $\mathcal{N}_M \sim \mathcal{O}$ such that for any $V \sim \mathcal{N}_M$, $M(V)$ is not $\mathcal{O}$-legal.

Theorem 7 *For any $M \in S(X) - DP(\mathcal{O})$, there exists a CP-net $\mathcal{N}_M \sim \mathcal{O}$ s.t. M is not $(\mathcal{N}_M, \mathcal{O})$-legal.*

Order-changing permutations

In this section, we consider the case where P and $M(P)$ are compatible with different orders. The study of this case is motivated by the definition of strongly decomposable rules. Fortunately, nearly all results in the last subsection can be extended to this case (however, the proofs are much harder). We first define an interesting property describing the relation between two orders. We say two orders are *similar* if the number of elements of the same ranked subdomains in the two orders are the same.

Definition 11 *Two orders* $\mathcal{O} = \mathbf{x}_{\sigma(1)} > \ldots > \mathbf{x}_{\sigma(p)}$, $\mathcal{O}' = \mathbf{x}_{\gamma(1)} > \ldots > \mathbf{x}_{\gamma(p)}$ *are said to be* similar, *if for all* $i \leq p$, $|D_{\sigma(i)}| = |D_{\gamma(i)}|$.

We observed that if a permutation M can always transform a CP-net compatible with $\mathcal{O}$ to another CP-net compatible with $\mathcal{O}'$, then $\mathcal{O}$ and $\mathcal{O}'$ are similar.

Theorem 8 *Given two orders* $\mathcal{O}, \mathcal{O}'$ *and* $M \in S(\mathcal{X})$, *if* M *is* $(\mathcal{N}, \mathcal{O}')$*-legal for all* $\mathcal{N} \sim \mathcal{O}$, *then* $\mathcal{O}'$ *must be similar to* $\mathcal{O}$.

We write $D_i = \{0_i, \ldots, (|D_i| - 1)_i\}$. When $|D_i| = |D_j|$, we define a standard mapping $f_{i,j}$ from D_i to D_j s.t. $f_{i,j}(k_i) = k_j$ for any $k \leq |D_i| - 1$. These standard permutations only exchange the names of elements in D_i and D_j. For example, when $D_1 = \{0_1, 1_1\}$ and $D_2 = \{0_2, 1_2\}$, then $f_{1,2}(0_1) = 0_2, f_{1,2}(1_1) = 1_2$. Now we are able to define such order-changing permutations.

Definition 12 *For any two similar orders* $\mathcal{O}$ *and* $\mathcal{O}'$, *define an* $(\mathcal{O}, \mathcal{O}')$*-induced permutation* $M_{\mathcal{O},\mathcal{O}'}$ *over* $\mathcal{X}$ *s.t. for any* $d_{\sigma(i)} \in D_{\sigma(i)}$

$$\begin{aligned}&M_{\mathcal{O},\mathcal{O}'}(d_{\sigma(1)}, \ldots, d_{\sigma(p)})\\=&(f_{\sigma(1),\gamma(1)}(d_{\sigma(1)}), \ldots, f_{\sigma(p),\gamma(p)}(d_{\sigma(p)})).\end{aligned}$$

Again we are concerned with the effect of $M_{\mathcal{O},\mathcal{O}'}$ on CP-nets. The induced permutation $P_{\mathcal{O},\mathcal{O}'}$ from $CP(\mathcal{O})$ to $CP(\mathcal{O}')$ is defined as follows. It only changes the name of the variables in the CP-net, namely changing $\mathbf{x}_{\sigma(i)}$ to $\mathbf{x}_{\gamma(i)}$.

Definition 13 *Given any two similar orders* $\mathcal{O}$ *and* $\mathcal{O}'$, *define an* $(\mathcal{O}, \mathcal{O}')$*-induced permutation* $P_{\mathcal{O},\mathcal{O}'}$ *from* $CP(\mathcal{O})$ *to* $CP(\mathcal{O}')$ *s.t. for any* $\mathcal{N} \in CP(\mathcal{O})$,

$$\begin{aligned}&\mathbf{x}_{\sigma(1)} = d_1 \ldots \mathbf{x}_{\sigma(i)} = d_i : x \succ_{\mathcal{N}} y\\ \Rightarrow&\mathbf{x}_{\gamma(1)} = f_{\sigma(1),\gamma(1)}(d_1) \ldots \mathbf{x}_{\gamma(i)} = f_{\sigma(i),\gamma(i)}(d_i) :\\ &f_{\sigma(i+1),\gamma(i+1)}(x) \succ_{P_{\mathcal{O},\mathcal{O}'}(\mathcal{N})} f_{\sigma(i+1),\gamma(i+1)}(y).\end{aligned}$$

Denote $DP(\mathcal{O}') \cdot M_{\mathcal{O},\mathcal{O}'} = \{M \cdot M_{\mathcal{O},\mathcal{O}'} : M \in DP(\mathcal{O}')\}$, where $M \cdot M_{\mathcal{O},\mathcal{O}'}$ is a permutation on $\mathcal{X}$ s.t. $M \cdot M_{\mathcal{O},\mathcal{O}'}(V) = M(M_{\mathcal{O},\mathcal{O}'}(V))$. We then present the order-changing version of Theorem 5, Theorem 6, and Theorem 7.

Theorem 9 *For any* $M \in DP(\mathcal{O}') \cdot M_{\mathcal{O},\mathcal{O}'}$ *and any CP-net* $\mathcal{N}$ *compatible with* $\mathcal{O}$, *if a vote* V *extends* $\mathcal{N}$ *and* $M(V)$ *is* $\mathcal{O}'$*-legal, then* $M(V)$ *extends* $f_{\mathcal{O}'}(M \cdot M_{\mathcal{O}',\mathcal{O}})(P_{\mathcal{O},\mathcal{O}'}(\mathcal{N}))$.

Theorem 10 *Let* $c_1, \ldots, c_p$ *be neutral correspondences on* $D_1, \ldots, D_p$ *respectively, such that* $(|D_i| = |D_j|) \Rightarrow (c_i = c_j)$. *For any* $\mathcal{O}$*-legal profile* P *and any* $M \in DP(\mathcal{O}') \cdot M_{\mathcal{O},\mathcal{O}'}$, *if* $M(P)$ *is* $\mathcal{O}'$*-legal, then*

$$M(Seq^{OI}(c_1, \ldots, c_p)(P)) = Seq^{OI}(c_1, \ldots, c_p)(M(P)).$$

Theorem 11 *For any* $M \in S(X) - DP(\mathcal{O}') \cdot M_{\mathcal{O},\mathcal{O}'}$, *there exists a CP-net* $\mathcal{N}_M \sim \mathcal{O}$ *s.t.* M *is not* $(\mathcal{N}_M, \mathcal{O}')$*-legal.*

Justifying decomposability

Since proving or refuting the neutrality of a decomposable rule is hard, we now relax the domain of decomposability and neutrality by applying them to a smaller domain $L = \{L_1, L_2, \ldots\}$ where $L_i \subseteq Legal(\mathcal{O})^i$. In order to keep the properties of legal profiles, we require L_i be approximately the set of all i-votes $\mathcal{O}$-legal profiles $Legal(\mathcal{O})^i$, i.e. with the number of voters i increases, L_i should occupy a large portion of $Legal(\mathcal{O})^i$. The next three concepts are defined to capture these ideas.

Definition 14 *Given* $\mathcal{X}$, *a countable sequence* $L = \{L_1, L_2, \ldots\}$ *is* nearly representative *for* $Legal(\mathcal{O})$ *if*

1. *For any* $i \in \mathbb{N}$, $L_i \subseteq Legal(\mathcal{O})^i$.
2. $\lim_{i\to\infty} \dfrac{|L_i|}{|Legal(\mathcal{O})|^i} = 1$.

Then we say a decomposable correspondence (rule) is *nearly neutral* if it is neutral on a sequence nearly representative for $Legal(\mathcal{O})$.

Definition 15 *A decomposable voting correspondence* $Seq(c_{\sigma(1)} \ldots, c_{\sigma(p)})$ *is* nearly neutral *for* $Legal(\mathcal{O})$ *if there exists a nearly representative sequence* L *for* $Legal(\mathcal{O})$ *such that for any* $i \in \mathbb{N}$, *any* $P \in L_i$, *and any permutation* $M \in S(\mathcal{X})$, *if* $M(P)$ *is* $\mathcal{O}$*-legal, then*

$$\begin{aligned}&M(Seq(c_{\sigma(1)}, \ldots, c_{\sigma(p)})(P))\\=&Seq(c_{\sigma(1)}, \ldots, c_{\sigma(p)})(M(P)).\end{aligned}$$

Obviously, if $Seq(c_{\sigma(1)} \ldots, c_{\sigma(p)})$ is neutral, then it is also nearly neutral, and when $L_i = Legal(\mathcal{O})^i$ for all i, nearly neutrality is equivalent to neutrality. Similarly, a nearly decomposable rule is a rule that coincides with a decomposable rule on a nearly representative sequence L for $Legal(\mathcal{O})$.

Definition 16 *A voting correspondence* c *on* $\mathcal{X}$ *is* nearly decomposable *if there exists* $c_1, \ldots, c_p$ *and a nearly representative sequence* L *for* $Legal(\mathcal{O})$ *s.t. for any* $i \in \mathbb{N}$ *and any* $P \in L_i$, $c(P) = Seq(c_{\sigma(1)} \ldots, c_{\sigma(p)})(P)$.

Now, we give an example of nearly representative sequence for $Legal(\mathcal{O})$. We say that a profile is $\mathcal{O}$*-universal* if its votes cover all possible CP-nets that are compatible with $\mathcal{O}$.

Definition 17 *An* $\mathcal{O}$*-legal profile* P *is* $\mathcal{O}$-universal *if for any CP-net* $\mathcal{N}$ *compatible with* $\mathcal{O}$, *there exists a vote* V *in* P *extending* $\mathcal{N}$.

Let us write $U_i(\mathcal{O}) = \{P : |P| = i, P \text{ is } \mathcal{O}\text{- universal}\}$; by simple calculations we can prove that $U(\mathcal{O}) = \{U_1(\mathcal{O}), \ldots\}$ is nearly representative for $Legal(\mathcal{O})$. Then we can give the main theorem of this section, which says that the sequential composition of any neutral correspondences is nearly neutral.

Theorem 12 *For any local neutral correspondences* $c_1, \ldots, c_p$, $Seq(c_1, \ldots, c_p)$ *is nearly neutral.*

Another interesting question is about the existence of neutral and nearly decomposable correspondences. The answer is affirmative. To see this, we define a correspondence C such that

$$C(P) = \begin{cases} Seq(c_1, \ldots, c_p)(P) & \text{If } P \text{ is } \mathcal{O}\text{-universal} \\ \mathcal{X} & \text{Otherwise} \end{cases}$$

For any non-universal $\mathcal{O}$-legal profile P and any permutation M, $M(P)$ cannot be universal by Theorem 7. So if $M(P)$ is $\mathcal{O}$-legal then $C(P) = C(M(P)) = M(C(P)) = \mathcal{X}$. For any universal profile P, from Theorem 7 we know that if $M \in S(\mathcal{X}) - DP(\mathcal{O})$ then $M(P)$ is not $\mathcal{O}$-legal, and from Theorem 6 we know that if $M \in DP(\mathcal{O})$ and $M(P)$ is $\mathcal{O}$-legal, then $C(P) = M(C(P))$. So C is neutral. Since U is a nearly representative sequence for $Legal(\mathcal{O})$, we know that C is nearly decomposable. This is summarized in the following theorem.

Theorem 13 *For any local neutral correspondences* $c_1, \ldots, c_p$, *there exists a neutral and nearly decomposable correspondence* C *on* $\mathcal{X}$ *s.t. for any universal profile* P , $C(P) = Seq(c_1, \ldots, c_p)(P)$.

Justifying strong decomposability

In this section, we study strong decomposability in a similar approximative framework.

Definition 18 *A countable sequence* $L = \{L_1, L_2, \ldots\}$ *is* nearly representative *for* $Legal(\mathcal{X})$ *if*

1. *For any* $i \in \mathbb{N}$, $L_i \subseteq Legal_i(\mathcal{X})$.
2. $\lim_{i\to\infty} \dfrac{|L_i|}{|Legal_i(\mathcal{X})|} = 1$.

A strongly decomposable voting correspondence $Seq^{OI}(c_1, \ldots, c_p)$ *is* nearly neutral for $Legal(\mathcal{X})$ *if it is neutral on some nearly representative sequence* L *for* $Legal(\mathcal{X})$. *A voting correspondence* c *on* $Legal(\mathcal{X})$ *is* nearly strongly decomposable, *if there exists* $c_1, \ldots, c_p$ *and a nearly representative sequence* L *for* $Legal(\mathcal{X})$ *s.t. for any* $i \in \mathbb{N}$ *and any* $P \in L_i$, $c(P) = Seq^{OI}(c_1, \ldots, c_p)(P)$.

Denote $U_i = \bigcup_{\mathcal{O}} U_i(\mathcal{O})$ the set of all universal profiles of i voters. We claim that $U = \{U_1, \ldots\}$ is nearly representative for $Legal(\mathcal{X})$. The main theorem of this section says that if a set of neutral local correspondences satisfy a necessary condition for their order-independent sequential composition to be neutral (see Theorem 4), then their order-independent sequential composition is nearly neutral.

Theorem 14 *For any neutral local correspondences* $c_1, \ldots, c_p$, *if* $(|D_i| = |D_j|) \Rightarrow (c_i = c_j)$, *then* $Seq^{OI}(c_1, \ldots, c_p)$ *is nearly neutral.*

Like Theorem 13, a similar construction leads to the the next theorem.

Theorem 15 *For any local neutral correspondences* $c_1, \ldots, c_p$, *if* $(|D_i| = |D_j|) \Rightarrow (c_i = c_j)$, *then there exists a neutral and nearly strong decomposable correspondence* C *on* $Legal(\mathcal{X})$ *such that for any universal profile* P,

$$C(P) = Seq^{OI}(c_1, \ldots, c_p)(P).$$

Conclusion and future work

To define the sequential composition of local voting rules without the ordering over attributes being fixed from the beginning, we introduced order-independent sequential composition and strong decomposability. We studied the properties of this new definition of decomposability. We studied to which extent some of the most relevant properties of voting rules can be lifted from local rules to their sequential composition. The most interesting of these properties is neutrality; in order to study neutrality of the composition of local voting rules, we first explored the properties of order-preserving and order-changing permutations, then we introduced the notions of near-decomposability and near-neutrality, in order to define an approximative framework to study the neutrality and (strong) decomposability on a large set of legal profiles. These results lead to the conclusion that the neutrality of local rules can always be nearly lifted to their (safe) sequential composition.

We plan to study further the properties of strong decomposability, especially the existence of neutral strong decomposable correspondences. Lastly, our order-independent compositions of local voting rules can be a solution to multiple election paradoxes (Brams, Kilgour, & Zwicker 1998) or simultaneous referenda (Lacy & Niou 2000). Separability allows for escaping these paradoxes; however, it is is a very demanding assumption. Our composition of local voting rules has a much wider range of applicability, and still allows to some extent to escape the paradoxes (see (Xia, Lang, & Ying 2007) for a preliminary study, with fixed-order sequential composition).

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