4

Computational Complexity of Unweighted Coalitional Manipulation

In this chapter, we will study the computational complexity of manipuation for three common voting rules. We will prove that the unweighted coalitional manipulation problem (Definition 3.1.2) is NP-complete for maximin (Section 4.1) and ranked pairs (Section 4.2), and we will give a polynomial-time algorithm for UCM for Bucklin (Section 4.3).

4.1 Manipulation for Maximin is NP-complete

In this section, we prove that the UCM problem for maximin is NP-complete. The proof uses a reduction from the TWO VERTEX DISJOINT PATHS IN DIRECTED ANTISYMMETRIC GRAPH problem, which is known to be NP-complete (Fortune et al., 1980).

Definition 4.1.1. *The* TWO VERTEX DISJOINT PATHS IN DIRECTED GRAPH *problem is defined as follows. We are given a directed graph G and two disjoint pairs of vertices (u, u') and (v, v'), where u, u', v, v' are all different from each other. We* *are asked whether there exist two directed paths* $u \to u_1 \to \ldots \to u_{k_1} \to u'$ *and* $v \to v_1 \to \ldots \to v_{k_2} \to v'$ *such that* $u, u', u_1, \ldots, u_{k_1}, v, v', v_1, \ldots, v_{k_2}$ *are all different from each other.*

Theorem 4.1.2. *The UCM problem for maximin is NP-complete, for any fixed number of manipulators* $n' \geqslant 2$.

Proof. It is easy to verify that the UCM problem for maximin is in NP. We now show that UCM is NP-hard, by giving a reduction from TWO VERTEX DISJOINT PATHS IN DIRECTED GRAPH.

Let the instance of TWO VERTEX DISJOINT PATHS IN DIRECTED GRAPH be denoted by $G = (\mathcal{V}, E)$, (u, u') and (v, v') where $\mathcal{V} = \{u, u', v, v', c_1, \ldots, c_{m-5}\}$. Without loss of generality, we assume that every vertex is reachable from u or v (otherwise, we can remove the vertex from the instance). We also assume that $(u, v') \notin E$ and $(v, u') \notin E$ (since such edges cannot be used in a solution). Let $G' = (\mathcal{V}, E \cup \{(v', u), (u', v)\})$, that is, G' is the graph obtained from G by adding (v', u) and (u', v).

We construct a UCM instance as follows.

Set of alternatives: $\mathcal{C} = \{c, u, u', v, v', c_1, \ldots, c_{m-5}\}$.

Alternative preferred by the manipulators: c.

Number of unweighted manipulators: any fixed number $n' \geqslant 2$.

Non-manipulators' profile: P^{NM} satisfying the following conditions:

1. For any $c' \neq c$, $D_{P^{NM}}(c, c') = -4n'$.

2. $D_{P^{NM}}(u, v') = D_{P^{NM}}(v, u') = -4n'$.

3. For any $(s, t) \in E$ such that $D_{P^{NM}}(t, s)$ is not defined above, we let $D_{P^{NM}}(t, s) = -2n' - 2$.

4. For any $s, t \in \mathcal{C}$ such that $D_{P^{NM}}(t, s)$ is not defined above, we let $|D_{P^{NM}}(t, s)| = 0$.

The existence of such a P^{NM}, whose size is polynomial in m, is guaranteed by Lemma 2.2.3.

We can assume without loss of generality that each manipulator ranks c first. Therefore, for any $c' \neq c$,

$$D_{P^{NM} \cup P^M}(c, c') = -3n' \tag{4.1}$$

We are now ready to show that there exists P^M such that Maximin$(P^{NM} \cup P^M) = c$ if and only if there exist two vertex disjoint paths from u to u' and from v to v' in G. First, we prove that if there exist such paths in G, then there exists a profile P^M for the manipulators such that Maximin$(P^{NM} \cup P^M) = c$.

Let $u \rightarrow u_1 \rightarrow \cdots \rightarrow u_{k_1} \rightarrow u'$ and $v \rightarrow v_1 \rightarrow \cdots \rightarrow v_{k_2} \rightarrow v'$ be two vertex disjoint paths. Further, let

$$\mathcal{V}' = \{u, u', v, v', u_1, \ldots, u_{k_1}, v_1, \ldots, v_{k_2}\} \ .$$

Then, because any vertex is reachable from u or v in G, there exists a connected subgraph G^* of G' (which still includes all the vertices) in which $u \rightarrow u_1 \rightarrow \cdots \rightarrow u_{k_1} \rightarrow u' \rightarrow v \rightarrow v_1 \rightarrow \cdots \rightarrow v_{k_2} \rightarrow v' \rightarrow u$ is the only cycle. In other words, such a subgraph G^* can obtained by possibly removing some of the edges of G'. Therefore, by arranging the vertices of $\mathcal{V} \backslash \mathcal{V}'$ according to the direction of the edges of G^*, we can obtain a linear order O over $\mathcal{V} \backslash \mathcal{V}'$ with the following property: for any $t \in \mathcal{V} \backslash \mathcal{V}'$, it holds that either

1. there exists $s \in \mathcal{V} \backslash \mathcal{V}'$ such that $s \succ_O t$ and $(s, t) \in E$, or
2. there exists $s \in \mathcal{V}'$ such that $(s, t) \in E$.

We define P^M by letting $n' - 1$ manipulators vote the following.

$$c \succ u \succ u_1 \succ \cdots \succ u_{k_1} \succ u' \succ v \succ v_1 \succ \ldots \succ v_{k_2} \succ v' \succ O$$

We also let the remaining manipulator vote the following.

$$c \succ v \succ v_1 \succ \cdots \succ v_{k_2} \succ v' \succ u \succ u_1 \succ \ldots \succ u_{k_1} \succ u' \succ O$$

Then, we have the following calculations:

$$\begin{aligned} D_{P^{NM} \cup P^M}(u, v') &= -4n' + (n' - 1) - 1 \\ &= -3n' - 2 < -3n' \quad , \\ \text{and} \quad D_{P^{NM} \cup P^M}(v, u') &= -4n' + 1 - (n' - 1) \\ &= -5n' + 2 < -3n' \quad . \end{aligned}$$

Moreover, for any $t \in \mathcal{C} \backslash \{c, u, v\}$, there exists $s \in \mathcal{C} \backslash \{c\}$ such that $(s, t) \in E$ and $D_{P^M}(t, s) = -n'$, which means that

$$\begin{aligned} D_{P^{NM} \cup P^M}(t, s) &= -2n' - 2 - n' = -3n' - 2 \\ &< -3n' \end{aligned}$$

It now follows from Equation (4.1) thatMaximin$(P^{NM} \cup P^M) = c$.

Next, we prove that if there exists a profile P^M for the manipulators such that Maximin$(P^{NM} \cup P^M) = c$, then there exist two vertex disjoint paths from u to u' and from v to v'.

We define a function $f : \mathcal{V} \to \mathcal{V}$ such that $D_{P^{NM} \cup P^M}(t, f(t)) < -3n'$. Indeed, such a function exists since Maximin$(P^{NM} \cup P^M) = c$. Hence, for any $t \neq c$ there must exist s such that

$$D_{P^{NM} \cup P^M}(t, s) < -3n'$$

Moreover, s must be a parent of t in G'. If there exists more than one such s, define $f(t)$ to be any one of them.

It follows that if $(t, f(t))$ is neither (u, v') or (v, u'), then $(f(t), t) \in E$ and $D_{P^M}(t, f(t)) = -n'$, which means that $f(t) \succ t$ in each vote of P^M; otherwise, if $(t, f(t))$ is (u, v') or (v, u'), then $D_{P^M}(t, f(t)) \leqslant n' - 2$, which means that $f(t) \succ t$ in at least one vote of P^M.

Now, since $|\mathcal{V}| = m - 1$ is finite, there must exist $l_1 < l_2 \leqslant m$ such that $f^{l_1}(u) = f^{l_2}(u)$. That is,

$$f^{l_1}(u) \to f^{l_1+1}(u) \to \cdots \to f^{l_2-1}(u) \to f^{l_2}(u)$$

is a cycle in G'. We assume that for any $l_1 \leqslant l_1' < l_2' < l_2$, $f^{l_1'}(u) \neq f^{l_2'}(u)$. Now we claim that (v', u) and (u', v) must be both in the cycle, because

1. if neither of them is in the cycle, then in each vote of P^M, we must have

$$f^{l_2}(u) \succ f^{l_2-1}(u) \succ f^{l_1}(u) = f^{l_2}(u) \quad ,$$

which contradicts the assumption that each vote is a linear order;

2. if exactly one of them is in the cycle—without loss of generality, $f^{l_1}(u) = v$, $f^{l_1+1}(u) = u'$—then in at least one of the votes of P^M, we must have

$$f^{l_2}(u) \succ f^{l_2-1}(u) \succ \ldots \succ f^{l_1}(u) = f^{l_2}(u) \quad ,$$

which contradicts the assumption that each vote is a linear order.

Without loss of generality, let us assume that $f^{l_1}(u) = u$, $f^{l_1+1}(u) = v'$, $f^{l_3}(u) = v$, $f^{l_3+1}(u) = u'$, where $l_3 \leqslant l_2 - 2$. We immediately obtain two vertex disjoint paths:

$$u = f^{l_1}(u) = f^{l_2}(u) \to f^{l_2-1}(u) \to \ldots \to f^{l_3+1}(u) = u' \quad ,$$

and $v = f^{l_3}(u) \to f^{l_3-1}(u) \to \ldots \to f^{l_1+1}(u) = v'$. Therefore, UCM for maximin is NP-complete. □

Notice that the NP-completeness of UCM implies the NP-hardness of UCO for maximin.

4.2 Manipulation for Ranked Pairs is NP-complete

In this section, we prove that UCM for ranked pairs is NP-complete (even for a single manipulator) by giving a reduction from 3SAT.

Definition 4.2.1. *The* 3SAT *problem is defined as follows: Given a set of variables* $X = \{x_1, \ldots, x_q\}$ *and a formula* $Q = Q_1 \wedge \ldots \wedge Q_t$ *such that*

1. *for all* $1 \leqslant i \leqslant t$, $Q_i = l_{i,1} \vee l_{i,2} \vee l_{i,3}$, *and*

2. *for all* $1 \leqslant i \leqslant t$ *and* $1 \leqslant j \leqslant 3$, $l_{i,j}$ *is either a variable* $x \in X$, *or the negation of a variable (i.e.,*$\neg x$ *where* $x \in X$*),*

we are asked whether the variables can be set to true or false so that Q *is true.*

Theorem 4.2.2. *The UCM problem for ranked pairs is NP-complete, even when there is only one manipulator.*

Proof. It is easy to verify that UCM for ranked pairs are in NP. We first prove that UCM is NP-complete. Given an instance of 3SAT, we construct a UCM instance as follows. Without loss of generality, we assume that for any variable x, x and $\neg x$ appears in at least one clause, and none of the clauses contain both x and $\neg x$.

Set of alternatives: $\mathcal{C} = \{c, Q_1, \ldots, Q_t, Q'_1, \ldots, Q'_t\} \bigcup \{x_1, \ldots, x_q, \neg x_1, \ldots, \neg x_q\}$ $\bigcup \{Q_{l_{1,1}}, Q_{l_{1,2}}, Q_{l_{1,3}}, \ldots, Q_{l_{t,1}}, Q_{l_{t,2}}, Q_{l_{t,3}}\} \bigcup \{Q_{\neg l_{1,1}}, Q_{\neg l_{1,2}}, Q_{\neg l_{1,3}}, \ldots, Q_{\neg l_{t,1}}, Q_{\neg l_{t,2}}, Q_{\neg l_{t,3}}\}$.

Alternative preferred by the manipulator: c.

Number of unweighted manipulators: $n' = 1$.

Tie-breaking mechanism: We recall that in ranked pairs, we first use the parallel-universe tie-breaking to select multiple winners, then use a fixed-order tie-breaking mechanism the select the unique winner. In the fixed-order tie-breaking, c is ranked in the bottom position.

Non-manipulators' profile: P^{NM} satisfying the following conditions.

1. For any $i \leqslant t$, $D_{PNM}(c, Q_i) = 30$,$D_{PNM}(Q'_i, c) = 20$; for any $x \in \mathcal{C} \backslash \{Q_i, Q'_i : 1 \leqslant i \leqslant t\}$, $D_{PNM}(c, x) = 10$.

2. For any $j \leqslant q$, $D_{PNM}(x_j, \neg x_j) = 20$.

3. For any $i \leqslant t, j \leqslant 3$,

   – if $l_{i,j} = x_k$ for some $k \leqslant q$, then $D_{PNM}(Q_i, Q^i_{x_k}) = 30$, $D_{PNM}(Q^i_{x_k}, x_k) = 30$, $D_{PNM}(\neg x_k, Q^i_{\neg x_k}) = 30$, $D_{PNM}(Q^i_{\neg x_k}, Q'_i) = 30$;

   – if $l_{i,j} = \neg x_k$ for some $k \leqslant q$, then $D_{PNM}(Q_i, Q^i_{\neg x_k}) = 30$, $D_{PNM}(Q^i_{x_k}, x_k) = 30$, $D_{PNM}(\neg x_k, Q^i_{\neg x_k}) = 30$, $D_{PNM}(Q^i_{x_k}, Q'_i) = 30$, $D_{PNM}(Q^i_{\neg x_k}, Q^i_{x_k}) = 20$.

4. For any $x, y \in \mathcal{C}$, if $D_{PNM}(x, y)$ is not defined in the above steps, then $D_{PNM}(x, y) = 0$.

For example, when $Q_1 = x_1 \vee \neg x_2 \vee x_3$, D_{PNM} is illustrated in Figure 4.1.

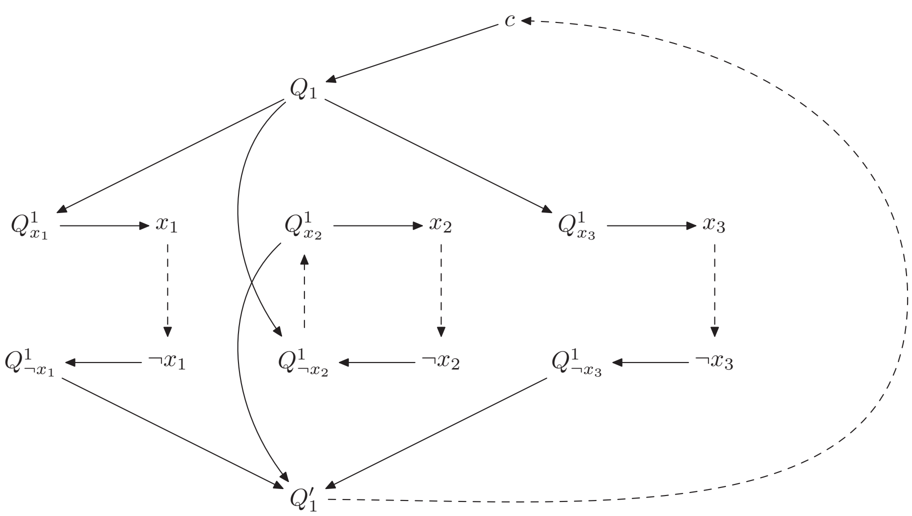

FIGURE 4.1: D_{PNM} for $Q_1 = x_1 \vee \neg x_2 \vee x_3$.

In Figure 4.1, for any vertices v_1, v_2, if there is a solid edge from v_1 to v_2, then $D_{PNM}(v_1, v_2) = 30$; if there is a dashed edge from v_1 to v_2, then $D_{PNM}(v_1, v_2) = 20$;

if there is no edge between v_1 and v_2 and $v_1 \neq c$, $v_2 \neq c$, then $D_{P^{NM}}(v_1, v_2) = 0$; for any x such that there is no edge between c and x, $D_{P^{NM}}(c, x) = 10$.

The existence of such a P^{NM} is guaranteed by Lemma 2.2.3, and the size of P^{NM} is in polynomial in t and q.

First, we prove that if there exists an assignment v of truth values to X so that Q is satisfied, then there exists a vote R_M for the manipulator such that $\text{RP}(P^{NM} \cup \{R_M\}) = c$. We construct R_M as follows.

- Let c be on the top of R_M.
- For any $k \leqslant q$, if $v(x_k) = \top$ (that is, x_k is true), then $x_k \succ_{R_M} \neg x_k$, and for any $i \leqslant t, j \leqslant 3$ such that $l_{i,j} = \neg x_k$, let $Q^i_{x_k} \succ_{R_M} Q^i_{\neg x_k}$.
- For any $k \leqslant q$, if $v(x_k) = \bot$ (that is, x_k is false), then $\neg x_k \succ_{R_M} x_k$, and for any $i \leqslant t, j \leqslant 3$ such that $l_{i,j} = \neg x_k$, let $Q^i_{\neg x_k} \succ_{R_M} Q^i_{x_k}$.
- The remaining pairs of alternatives are ranked arbitrarily.

If $x_k = \top$, then $D_{P^{NM} \cup \{R_M\}}(x_k, \neg x_k) = 21$, and for any $i \leqslant t, j \leqslant 3$ such that $l_{i,j} = \neg x_k$, $D_{P^{NM} \cup \{R_M\}}(Q^i_{\neg x_k}, Q^i_{x_k}) = 19$. It follows that no matter how ties are broken when applying ranked pairs to $P^{NM} \cup \{R_M\}$, if $x_k = \top$, then $x_k \succ \neg x_k$ in the final ranking. This is because for any $l_{i,j} = \neg x_k$, $D_{P^{NM} \cup \{R_M\}}(Q^i_{\neg x_k}, Q^i_{x_k}) = 19 < 21 = D_{P^{NM} \cup \{R_M\}}(x_k, \neg x_k)$, which means that before trying to fix $x_k \succ \neg x_k$, there is no directed path from $\neg x_k$ to x_k.

Similarly if $x_k = \bot$, then $D_{P^{NM} \cup \{R_M\}}(x_k, \neg x_k) = 19$, and for any $i \leqslant t, j \leqslant 3$ such that $l_{i,j} = \neg x_k$, $D_{P^{NM} \cup \{R_M\}}(Q^i_{\neg x_k}, Q^i_{x_k}) = 21$. It follows that if $x_k = \bot$, then $\neg x_k \succ x_k$, and for any $i \leqslant t, j \leqslant 3$ such that $l_{i,j} = \neg x_k$, $Q^i_{\neg x_k} \succ Q^i_{x_k}$ in the final ranking. This is because $Q^i_{\neg x_k} \succ Q^i_{x_k}$ will be fixed before $x_k \succ \neg x_k$.

Because Q is satisfied under v, for each clause Q_i, at least one of its three literals is true under v. Without loss of generality, we assume $v(l_{i,1}) = \top$. If $l_{i,1} = x_k$, then before trying to add $Q'_i \succ c$, the directed path $c \to Q_i \to Q_{x_k} \to x_k \to \neg x_k \to Q_{\neg x_k} \to Q'_i$ has already been fixed. Therefore, $c \succ Q'_i$ in the final ranking, which means that for any alternatives x in $\mathcal{C} \backslash \{c, Q_1, \ldots, Q_t, Q'_1, \ldots, Q'_t\}$, $c \succ x$ in the final ranking because $D_{P^{NM} \cup \{R_M\}}(c, x) > 0$. Hence, c is the unique winner of $P^{NM} \cup \{R_M\}$ under ranked pairs.

Next, we prove that if there exists a vote R_M for the manipulator such that $\text{RP}(P^{NM} \cup \{R_M\}) = c$, then there exists an assignment v of truth values to X such that Q is satisfied. We construct the assignment v so that $v(x_k) = \top$ if and only if $x_k \succ_{R_M} \neg x_k$, and $v(x_k) = \bot$ if and only if $\neg x_k \succ_{R_M} x_k$. We claim that $v(Q) = \top$. If, on the contrary, $v(Q) = \bot$, then there exists a clause (Q_1, without loss of generality) such that $v(Q_1) = \bot$. We now construct a way to fix the pairwise rankings such that c is not the winner for ranked pairs, as follows. For any $j \leqslant 3$, if there exists $k \leqslant q$ such that $l_{i,j} = \neg x_k$, then $x_k \succ_{R_M} \neg x_k$ because $v(\neg x_k) = \bot$. Therefore, $D_{P^{NM} \cup R_M}(x_k, \neg x_k) = 21$. Then, after trying to add all pairs $x \succ x'$ such that $D_{P^{NM} \cup R_M}(x, x') > 21$ (that is, all solid directed edges in Figure 4.1), it follows that $x_k \succ \neg x_k$ can be added to the final ranking. We choose to add $x_k \succ \neg x_k$ first, which means that $Q^1_{x_k} \succ Q^1_{\neg x_k}$ in the final ranking (otherwise, we have $Q^1_{\neg x_k} \succ Q^1_{x_k} \succ x_k \succ \neg x_k \succ Q^1_{\neg x_k}$, which is a contradiction).

For any $j \leqslant 3$, if there exists $k \leqslant q$ such that $l_{i,j} = x_k$, then $\neg x_k \succ_{R_M} x_k$ because $v(x_k) = \bot$. Therefore, $D_{P^{NM} \cup R_M}(x_k, \neg x_k) = 19$. We note that after trying to add all pairs $x \succ x'$ such that $D_{P^{NM} \cup R_M}(x, x') > 19$, $Q^1_{x_k} \not\succ Q^1_{\neg x_k}$. We recall that for any $j \leqslant 3$, if there exists $k \leqslant q$ such that $l_{i,j} = \neg x_k$, then $Q^1_{\neg x_k} \not\succ Q^1_{x_k}$. Hence, it follows that $Q'_1 \succ c$ is consistent with all pairwise rankings added so far. Then, since $D_{P^{NM} \cup R_M}(Q'_1, c) \geqslant 19$, if $Q'_1 \succ c$ has not been added, we choose to add it first of all pairwise rankings of alternatives $x \succ x'$ such that $D_{P^{NM} \cup R_M}(x, x') = 19$, which means that $Q'_1 \succ c$ in the final ranking—in other words, c is not at the top in the final ranking. Therefore, c is not the unique winner, which contradicts the assumption that $\mathrm{RP}(P^{NM} \cup \{R_M\}) = c$. □

Similarly, we can prove that when k is a constant greater than one, UCM for ranked pairs remain NP-complete.

Theorem 4.2.3. *The UCM problem for ranked pairs is NP-complete, for any fixed number of manipulators $n' \geqslant 2$.*

Proof. The proof is similar to that of Theorem 4.2.2. We let P^{NM} satisfy the following conditions.

1. For any $i \leqslant t$, $D_{P^{NM}}(c, Q_i) = 30n'$,$D_{P^{NM}}(Q'_i, c) = 22n' - 2$; for any $x \in \mathcal{C} \backslash \{Q_i, Q'_i : 1 \leqslant i \leqslant t\}$, $D_{P^{NM}}(c, x) = 10n'$.
2. For any $j \leqslant q$, $D_{P^{NM}}(x_j, \neg x_j) = 22n' - 2$.
3. For any $i \leqslant t, j \leqslant 3$, if $l_{i,j} = x$, then $D_{P^{NM}}(Q_i, Q^i_x) = 30n'$, $D_{P^{NM}}(Q^i_x, x) = 30n'$, $D_{P^{NM}}(\neg x, Q^i_{\neg x}) = 30n'$, $D_{P^{NM}}(Q^i_{\neg x}, Q'_i) = 30n'$; if $l_{i,j} = \neg x$, then $D_{P^{NM}}(Q_i, Q^i_{\neg x}) = 30n'$, $D_{P^{NM}}(Q^i_x, x) = 30n'$, $D_{P^{NM}}(\neg x, Q^i_{\neg x}) = 30n'$, $D_{P^{NM}}(Q^i_x, Q'_i) = 30n'$, $D_{P^{NM}}(Q^i_{\neg x}, Q^i_x) = 20n'$.
4. For any $y, y' \in \mathcal{C}$, if $D_{P^{NM}}(y, y')$ is not defined in the above steps, then $D_{P^{NM}}(y, y') = 0$.

First, if there exists an assignment v of truth values to X so that Q is satisfied, then we let R_M be defined as in the proof for Theorem 4.2.2. It follows that $\mathrm{RP}(P^{NM} \cup \{n'R_M\}) = c$ (all the manipulators can vote R_M).

Next, if there exists a profile P^M for the manipulators such that $\mathrm{RP}(P^{NM} \cup P^M) = c$, then we construct the assignment v so that $v(x) = \top$ if $x \succ_V \neg x$ for all $V \in P^M$, and $v(x) = \perp$ if $\neg x \succ_V x$ for all $V \in P^M$; the values of all the other variables are assigned arbitrarily. Then by similar reasoning as in the proof for Theorem 4.2.2, we know that Q is satisfied under v. □

4.3 A Polynomial-time Algorithm for Manipulation for Bucklin

In this section, we present a polynomial-time algorithm for the UCM problem for Bucklin.

For any alternative $x \in \mathcal{C}$, any natural number $d \in \mathbb{N}$, and any profile P, let $B(x, d, P)$ denote the number of times that x is ranked among the top d alternatives in P. The idea behind the algorithm is as follows. Let d_{min} denote the Bucklin score of x in P, that is, d_{min} is the minimal depth so that the favorite alternative c is ranked among the top d_{min} alternatives in more than half of the votes (when all of the manipulators rank c first). Then, we simply check if there is a way to assign the manipulators' votes so that none of the other alternatives is ranked among the top d_{min} alternatives in more than half of the votes. In other words, the order of the alternatives is not crucial, only their membership in the set of d_{min} top-ranked alternatives is relevant.

Algorithm 1.

Input. A UCM instance $(Bucklin, P^{NM}, c, n')$, where $\mathcal{C} = \{c, c_1, \ldots, c_{m-1}\}$.[1]

Stage 0.

0.1 Calculate the Bucklin score d_{min} such that

$$B(c, d_{min}, P^{NM}) + n' > \frac{1}{2}(|P^{NM}| + n')$$

0.2 If there exists $c' \in C$, $c' \neq c$ such that

$$B(c', d_{min}, P^{NM}) > \frac{1}{2}(|P^{NM}| + n') \quad , \tag{4.2}$$

then output that there is no successful manipulation.

[1] This algorithm works for the fixed-order tie-breaking mechanism where c is ranked in the bottom position. Similar algorithms can be designed for other fixed-order tie-breaking mechanisms.

Aside. Notice that d_{min} is defined under the assumption that all the manipulators rank c first. Consider an alternative $c' \neq c$ that satisfies the condition in Equation (4.2). Such an alternative is ranked in the top d_{min} positions of half the votes $P^{NM} \cup P^{M}$, regardless of P^{M}. Hence, c cannot be a unique winner.

Stage 1.

1.1 For every $c' \in C \backslash \{c\}$, let

$$d_{c'} = \left\lfloor \frac{1}{2}(|P^{NM}| + n') \right\rfloor - B(c', d_{min}, P^{NM}) \quad ,$$

and let $k_{c'} = \min\{d_{c'}, n'\}$.

1.2 If

$$\sum_{c' \neq c} k_{c'} < (d_{min} - 1)n' \quad , \tag{4.3}$$

then output that there is no successful manipulation.

Aside. $k_{c'}$ is the number of times that we can place c' in the first d_{min} positions of the votes of P^{M}, without compromising the victory of c. In particular, $k_{c'}$ cannot be greater than n'.

Notice that there are exactly $(d_{min} - 1)n'$ problematic positions to fill, since c is ranked first by all the manipulators. Now, if the condition in Equation (4.3) is satisfied, for any P^{M} there must be an alternative c' that appears too many times in the first d_{min} positions, that is, $k_{c'} < B(c', d_{min}, P^{M})$. Since $B(c', d_{min}, P^{M}) \leqslant n'$, we have in particular that $k_{c'} < n'$, hence it must hold that $k_{c'} = d_{c'}$. It follows that

$$\begin{aligned}
&B(c', d_{min}, P^{NM} \cup P^{M}) \\
=&B(c', d_{min}, P^{NM}) + B(c', d_{min}, P^{M}) \\
>&B(c', d_{min}, P^{NM}) + d_{c'} \\
=&\left\lfloor \frac{1}{2}(|P^{NM}| + n') \right\rfloor
\end{aligned}$$

Therefore, c cannot be a unique winner.

Stage 2. Construct P^M by assigning the alternatives to the first d_{min} positions of the votes in a way that for every $t = 1, \ldots, m-1$,

$$B(c_t, d_{min}, P^M) \leqslant k_{c_t} \tag{4.4}$$

Complete the rest of the votes arbitrarily. Return P^M as a successful manipulation.

Aside. Given that (4.3) does not hold, it is clearly possible to construct P^M such that (4.4) holds for every $c' \neq c$. Moreover, this can be done in polynomial time, e.g., by enumerating the alternatives and placing each alternative in the next position in $k_{c'}$ of the votes of the manipulators, until the crucial positions are filled.

Now, for every $t = 1, \ldots, m-1$ it holds that

$$\begin{aligned} B(c_t, d_{min}, P^{NM} \cup P^M) &\leqslant B(c_t, d_{min}, P^{NM}) + k_{c_t} \\ &\leqslant \frac{1}{2}(|NM| + n') \quad , \end{aligned}$$

which implies that $Bucklin(P^{NM} \cup P^M) = c$.

We have obtained the following result.

Theorem 4.3.1. *Algorithm 1 correctly decides the UCM problem in polynomial time.*

It is easy to see that the tractability of UCM for Bucklin implies that UCO can be solved in polynomial time as well.

4.4 Summary

In this chapter, we investigated the computational complexity of the UCM and UCO problems for the maximin, ranked pairs, and Bucklin rules. The UCM problem is NP-complete under the maximin rule for any fixed number (at least two) of manipulators. The UCM problem is also NP-complete under the ranked pairs rule; in this case, the hardness holds even if there is only a single manipulator, similarly to second-order Copeland (Bartholdi et al., 1989a) and STV (Bartholdi and Orlin, 1991). Finally, we gave a polynomial-time algorithm for the UCM problem under the Bucklin rule.

It should be noted that all of our NP-hardness results, as well as the ones mentioned in the introduction, are *worst-case* results. Hence, there may still be an efficient algorithm that can find a beneficial manipulation for *most* instances. Indeed, nearly a dozen recent papers suggest that finding manipulations is easy with respect to some typical distributions on preference profiles. We will see some of them in the next chapter.