# Settling the neutrality and efficiency of decomposable voting rules and correspondences

Lirong Xia
lxia@cs.duke.edu

Jérôme Lang
lang@irit.fr

Mingsheng Ying
yingmsh@mail.tsinghua.edu.cn

**Abstract**

In this paper, we address the problem of the existence of neutral or efficient decomposable voting rules and correspondences on combinatorial domains, by extending the impossibility theorems about the neutrality and efficiency of seat-by-seat voting rules in a recent paper of Benoit and Kornhauser. Our contributions are the following: we identify the set of neutral rules on two-binary-issue combinatorial domains, which is not covered by their work; and we prove the impossibility theorem for voting correspondences (while Benoit and Kornhauser only consider voting rules). Together with previous work, our work thus completely solves the problem of the existence of neutral or efficient decomposable voting correspondences.

## 1 Introduction

With more and more applications of voting theory in computer science, researchers are exposed to the computational problems—the size of the preference structure that models the voters' preferences increases exponentially to the size of the candidate set. To solve this problem, researchers are interested in voting processes with incomplete information, namely some kinds of compact representation. In many real-life instances, the candidates can be described by several attributes (or issues), thus the whole candidate set can be represented by a Cartesian product of all attribute domains. Casting votes and aggregating them on such a combinatorial domain thus arouses more and more interests [3, 1].

Lots of work have been done on the case that preferences are *separable*, which is, the voters' preference can be safely projected onto each issue independently, in order to avoid *multiple election paradoxes* (see [2] [7], [11]). Then, aggregation can be done in a issue-wise manner, called *seat-by-seat* voting. However, the separability of voting profiles are pointed out to be too demanding because the separable profiles constitute a tiny proportion of all possible profiles (see [9]).

To capture more instances on a structured domain whiling keeping a compact representation, Lang [12] suggested a *sequential voting process* based on the conditional preferences of voters modeled by a concept recently developed in AI—the CP-nets [5]. Such voting process defines *decomposable voting rules* which can also avoid multiple election paradoxes [13]. The seat-by-seat voting is a special case of this sequential voting.

There have been some important problems concerning seat-by-seat voting rules, for example the neutrality and efficiency. This problem has been well studied, and recently Benoit and Kornhauser proved two impossibility theorems [4] to partly solve it. They showed that if the combinatorial domain is not composed of two binary issues, then the only efficient seat-by-seat rule is the dictatorship; if the combinatorial domain consists at least three issues, then the only local-efficient and neutral seat-by-seat voting rule is the dictatorship. In [14] there are some results on the neutrality that slightly relaxes the domain restriction. However, there are still three cases not covered by these two theorems. First, there was no result about the combinatorial domain composed of two binary issues. Second, the existence of neutral seat-by-seat rule on the domain composed of two issues (not necessarily binary) was not clear. And third, they didn't discuss seat-by-seat correspondence.

The first question has already been answered in [13], in which the authors showed that the sequential composition of two majority rules on the domain composed of two binary issues satisfies neutrality and efficiency. This theorem can be easily applied to seat-by-seat rules.

We solved the second problem in this paper by an impossibility theorem. Our theorems even show that the local efficiency condition can be removed from the theorems in [4] while keeping the result nearly unchanged.

The third problem is interesting, especially on the neutrality side. This is mainly because that when there are odd number of voters, usually no voting rule can be neutral, but there are lots of nontrivial neutral correspondences. Lang [12] conjectured that there is no non-trivial neutral decomposable voting correspondence in many cases. In this paper we will prove that if a seat-by-seat correspondence satisfies neutrality or efficiency, then it must be a dictatorship or the trivial one that selects all candidates regardless of the profile. This gives a justification of Lang's conjecture.

The paper is structured as follows. In the next section we recall some basic definitions of CP-nets, sequential voting process, and seat-by-seat voting rules[1]. We present our main theorems on the neutrality and efficiency of seat-by-seat voting correspondences in Section 3, and conclude the paper in Section 4. Proofs of the theorems are deferred to the Appendix.

# 2 Preliminaries

## 2.1 Voting rules and correspondences

We start by recalling briefly some necessary background on voting rules and correspondences.

Let $\mathcal{X}$ be a finite set of *alternatives*. A *(strict) preference relation* on $\mathcal{X}$ is a strict order (an irreflexive, asymmetric and transitive binary relation). A *vote* $V$ is a linear preference relation on $\mathcal{X}$, i.e., a complete strict order (for any $x$ and $y \neq x$, either $x \succ y$ or $y \succ x$ holds). We often note $x \succ_V x'$ instead of $V(x,x')$. For any linear preference $V$, we write $(V)_l$ to denote the $l$-th ranked candidate in $V$. When $i = 1$, $(V)_1$ is called the *top* $V$, denoted by $top(V)$. When $i = |\mathcal{X}|$, $(V)_{|\mathcal{X}|}$ is called the *bottom* of $V$, denoted by $bot(V)$.

Let $\{1,...,N\}$ be a finite set of *voters*. A $N$-voter profile w.r.t. $\mathcal{X}$ is a collection of $N$ individual linear[2] preference relations over $\mathcal{X}$: $P = (V_1,...,V_N)$.

Let $P_{N,\mathcal{X}}$ be the set of all $N$-voter preference profiles for the set of candidates $\mathcal{X}$. A *voting correspondence* $C : P_{N,\mathcal{X}} \to 2^{\mathcal{X}} \setminus \{\emptyset\}$ maps each preference profile $P$ of $P_{N,\mathcal{X}}$ into a nonempty subset $C(P)$ of $\mathcal{X}$. A *voting rule* $r : P_{N,\mathcal{X}} \to \mathcal{X}$ maps each preference profile $P$ of $P_{N,\mathcal{X}}$ into a single candidate $r(P)$. The correspondence that elects the candidates that are ranked first by the largest number of voters is the

[1]Part of this section is copied from our submitted journal paper, it is included just for the convenience of the readers.

[2]For the sake of simplicity, in this paper we do not allow for indifferences; however most of the results in this paper would be easy to extend if indifferences are allowed.

*plurality* correspondence. When there are only two candidates $\{x,y\}$, the *majority* correspondence $maj$ is defined by $maj(P) = \{x\}$ (resp. $\{y\}$ if more voters in $P$ prefer $x$ to $y$ (resp. $y$ to $x$), and $maj(P) = \{x,y\}$ in case of tie.

We recall a few important properties that voting rules and correspondences may (or may not) satisfy. A voting rule satisfies

- **neutrality** if for any profile $P$ and any permutation $M$ on candidates, $r(M(P)) = M(r(P))$.
- **efficiency** if for any profile $P = (V_1,\ldots,V_N)$, there is no candidate $c$ s.t. $c >_{V_i} r(P)$ for all $i \leq N$.

A voting correspondence $C$ satisfies

- **neutrality** if for any profile $P$ and any permutation $M$ on candidates, $C(M(P)) = M(C(P))$.
- **efficiency** if for any profile $P = (V_1,\ldots,V_N)$, if whenever there exist two candidates $x,y$ such that $x >_{V_i} y$ for all $i \leq N$, we have: $y \in C(P)$ implies $x \in C(P)$.

A voting rule $r$ is a dictatorship (respectively an anti-dictatorship) if there exists a voter $j$ such that for any $N$-voter profile $P = (V_1,\ldots,V_N)$, $r(P) = top(V_j)$ (respectively $r(P) = bot(V_j)$). A voting correspondence $C$ is a dictatorship (respectively an anti-dictatorship) if there exists a voter $j$ such that for any $N$-voter profile $P = (V_1,\ldots,V_N)$, $C(P) = \{top(V_j)\}$ (respectively $C(P) = \{bot(V_j)\}$.

## 2.2 Preferences on multi-issue domains

Let $I = \{\mathbf{x}_1,\ldots,\mathbf{x}_p\}$ be a set of *issues*. For each $\mathbf{x}_i \in I$, $D_i$ is the finite *value domain* of $\mathbf{x}_i$. Without loss of generality, we assume $|D_i| \geq 2$ for every $i$. An issue $\mathbf{x}_i$ is *binary* if $|D_i| = 2$; in this case, we use the following two notations: $D_i = \{x_i,\overline{x_i}\}$ and $D_i = \{1_i,0_i\}$. (Note the difference between the issue $\mathbf{x}_i$ and the value $x_i$.) If $X = \{\mathbf{x}_{i_1},\ldots,\mathbf{x}_{i_m}\} \subseteq I$, with $i_1 < \ldots < i_p$, then $D_X$ denotes $D_{i_1} \times \ldots \times D_{i_m}$.

$\mathcal{X} = D_1 \times \ldots \times D_p$ is the set of all *alternatives* (or *candidates*). Elements of $\mathcal{X}$ are denoted by vectors $\vec{x},\vec{x}'$ etc. and represented by concatenating the values of the issues: for instance, if $I = \{\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3\}$, $x_1\overline{x_2}x_3$ assigns $x_1$ to $\mathbf{x}_1$, $\overline{x_2}$ to $\mathbf{x}_2$

and $x_3$ to $\mathbf{x}_3$. We allow concatenations of vectors of values: for instance, let $I = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5\}$, $Y = \{\mathbf{x}_1, \mathbf{x}_2\}$, $Z = \{\mathbf{x}_3, \mathbf{x}_4\}$, $\vec{y} = x_1\overline{x_2}$, $\vec{z} = \overline{x_3}x_4$, then $\vec{y}.\vec{z}.\overline{x_5}$ denotes the alternative $x_1\overline{x_2}\overline{x_3}x_4\overline{x_5}$.

Let $\{X, Y, Z\}$ be a partition of the set $I$ and $V$ a linear preference relation over $\mathcal{X} = D_I$. $X$ is *conditionally preferentially independent* of $Y$ given $Z$ (w.r.t. $V$) if and only if for all $\vec{x}_1, \vec{x}_2 \in D_X$, $\vec{y}_1, \vec{y}_2 \in D_Y$, $\vec{z} \in D_Z$,

$$\vec{x}_1.\vec{y}_1.\vec{z} \succ \vec{x}_2.\vec{y}_1.\vec{z} \text{ iff } \vec{x}_1.\vec{y}_2.\vec{z} \succ_V \vec{x}_2.\vec{y}_2.\vec{z}$$

Informally, $X$ is *conditionally preferentially independent* of $Y$ given $Z$ given a fixed value $\vec{z}$ of $Z$, the preference over the possible values of $X$ is independent from the value of $Y$. We use the notation $CPI_V(X, Y, Z)$ to denote that $X$ is conditionally preferentially independent of $Y$ given $Z$ (w.r.t. $V$). When $X$ is a singleton we simply note $CPI_V(\mathbf{x}, Y, Z)$ instead of $CPI_V(\{\mathbf{x}\}, Y, Z)$.

Conditional preferential independence originates in the literature of multiattribute decision theory [10]. Unlike probabilistic independence, it is a directed notion: $X$ may be independent of $Y$ given $Z$ without $Y$ being independent of $X$ given $Z$. Note that preferential independence is weaker than utility independence.

*Conditional preference networks*, or *CP-nets*, are a language for specifying preferences based on the notion of conditional preferential independence. They allow for eliciting preferences, and for storing them, as economically as possible. Formally, a *CP-net* $\mathcal{N}$ [5] over $I$ is a pair consisting of:

- a *directed acyclic*[3] *graph* $G = \langle I, E \rangle$ whose set of vertices is $I$ and set of edges $E$. For any $\mathbf{x} \in I$, $Par_G(\mathbf{x})$ denotes the set of parents of $\mathbf{x}$ in $G$, that is, $\{\mathbf{y} \in I \mid (\mathbf{y}, \mathbf{x}) \in E\}$.
- a collection of *conditional preference tables* $CPT(\mathbf{x}_i)$ for each $\mathbf{x}_i \in V$. Each conditional preference table $CPT(\mathbf{x}_i)$ associates a total order $\succ^i_{\vec{u}}$ with each instantiation $\vec{u}$ of $\mathbf{x}_i$'s parents $Par_G(\mathbf{x}_i) = U$.

The conditional preference statements contained in these tables are written with the following notation: $x_1\overline{x_2} : x_3 \succ \overline{x_3}$ means that when $\mathbf{x}_1 = x_1$ and $\mathbf{x}_2 = \bar{x}_2$

[3]CP-nets are also defined in the more general case where $G$ contains cycles. However, in this paper we do noe need to refer to this more general framework. Note that the acyclicity assumption is acyclic is usual (see [5, 6]), especially because allowing for possibly cyclic graphs makes the framework much less satisfactory from a computational point of view [5, 8].

then $\mathbf{x}_3 = x_3$ is preferred to $\mathbf{x}_3 = \overline{x_3}$.

A CP-net $\mathcal{N}$ induces a partial preference relation in the following way. For each $\mathbf{x}_i \in V$, let $Z = Par_G(\mathbf{x}_i)$ and $Y = \overline{\{\mathbf{x}_i\} \cup Z}$; then let $\succ_{\mathcal{N}}^{\mathbf{x}_i}$ be the *local relation* (to $\mathbf{x}_i$) defined by $\succ_{\mathcal{N}}^{\mathbf{x}_i} = \{(x_i.\vec{y}.\vec{z}, x_i'.\vec{y}.\vec{z}) \mid \vec{y} \in D_Y, \vec{z} \in D_Z, CPT(\mathbf{x}_i)$ contains $\vec{z} : x_i \succ x_i'\}$. Let $Prim(\mathcal{N}) = \bigcup\{\succ_{\mathcal{N}}^{\mathbf{x}_i} \mid \mathbf{x}_i \in I\}$ be the *primitive relation* induced by $\mathcal{N}$. The relation $\succ_{\mathcal{N}}$ is the transitive closure of $Prim(\mathcal{N})$. $\succ_{\mathcal{N}}$ is an irreflexive and asymmetric relation that possesses a dominating element [5][4]. *Note that the preference relation induced from a CP-net is generally not complete.* We say a linear preference $V$ extends $\mathcal{N}$, or $\succ_{\mathcal{N}}$, if $\succ_{\mathcal{N}} \subseteq V$, namely for any $\alpha, \beta \in \mathcal{X}$, $(\alpha \succ_{\mathcal{N}} \beta) \Rightarrow \alpha \succ_V \beta$.

**Example 1** *Let $\mathcal{N}$ be the CP-net whose graph G is depicted below. This graph means that the agent's preference over the values of* $\mathbf{x}$ *is unconditional, preference over the values of* $\mathbf{y}$ *(resp.* $\mathbf{z}$*) is fully determined given the value of* $\mathbf{x}$ *(resp. the values of* $\mathbf{x}$ *and* $\mathbf{y}$*).*

*Let the conditional preference tables be:*

| $CPT(\mathbf{x})$ | $CPT(\mathbf{y})$ | $CPT(\mathbf{z})$ |
|---|---|---|
| $x \succ \bar{x}$ | $x : y \succ \bar{y}$ | $xy : z \succ \bar{z}$ |
| | $\bar{x} : \bar{y} \succ y$ | $x\bar{y} : z \succ \bar{z}$ |
| | | $\bar{x}y : z \succ \bar{z}$ |
| | | $\bar{x}\bar{y} : \bar{z} \succ z$ |

*The local relations induced by $\mathcal{N}$ are*

$\succ_{\mathcal{N}}^{\mathbf{x}}$: $xyz \succ \bar{x}yz$, $xy\bar{z} \succ \bar{x}y\bar{z}$, $x\bar{y}z \succ \bar{x}\bar{y}z$, $x\bar{y}\bar{z} \succ \bar{x}\bar{y}\bar{z}$

$\succ_{\mathcal{N}}^{\mathbf{y}}$: $xyz \succ x\bar{y}z$, $xy\bar{z} \succ x\bar{y}\bar{z}$, $\bar{x}\bar{y}z \succ \bar{x}yz$, $\bar{x}\bar{y}\bar{z} \succ \bar{x}y\bar{z}$

$\succ_{\mathcal{N}}^{\mathbf{z}}$: $xyz \succ xy\bar{z}$, $x\bar{y}z \succ x\bar{y}\bar{z}$, $\bar{x}yz \succ \bar{x}y\bar{z}$, $\bar{x}\bar{y}\bar{z} \succ \bar{x}\bar{y}z$

*The preference relation $\succ_{\mathcal{N}}$ is the transitive closure of $\succ_{\mathcal{N}}^{\mathbf{x}} \cup \succ_{\mathcal{N}}^{\mathbf{y}} \cup \succ_{\mathcal{N}}^{\mathbf{z}}$, that is:*

$$xyz \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} x\bar{y}z \\ \\ xy\bar{z} \end{array} \begin{array}{c} \searrow \\ \nearrow \end{array} x\bar{y}\bar{z} \rightarrow \bar{x}\bar{y}\bar{z} \rightarrow \bar{x}\bar{y}z \rightarrow \bar{x}yz \rightarrow \bar{x}y\bar{z}$$

[4]The assumption that $G$ is acyclic is crucial for this statement to hold.

Let $G$ be an acyclic graph over $I$ and let $O = \mathbf{x}_1 > ... > \mathbf{x}_p$ be a linear order on $I$. $G$ is said to *follow* $O$ iff for every edge $(\mathbf{x}_i, \mathbf{x}_j)$ in $G$ we have $i < j$. A linear preference relation $\succ$ is said to be $O$-legal if and only if it is compatible with some acyclic graph $G$ following $O$. We denote by $Legal(O)$ the set of all $O$-legal linear preference relations.

Let $O = \mathbf{x}_1 > ... > \mathbf{x}_p$. Clearly, $\succ \in Legal(O)$ if and only if for all $i < p$, $\mathbf{x}_i$ is preferentially independent of $\{\mathbf{x}_{i+1}, ..., \mathbf{x}_p\}$ given $\{\mathbf{x}_1, ..., \mathbf{x}_{i-1}\}$ with respect to $\succ$. If $\succ \in Legal(O)$ then the *projection* of $\succ$ on $\mathbf{x}_i$ given $(x_1, \ldots, x_{i-1}) \in D_1 \times \ldots \times D_{i-1}$, denoted by $\succ^{\mathbf{x}_i|\mathbf{x}_1=x_1,\ldots,\mathbf{x}_{i-1}=x_{i-1}}$ (or equivalently $\succ|_{\mathbf{x}_i|\mathbf{x}_1=x_1,\ldots,\mathbf{x}_{i-1}=x_{i-1}}$), is the linear preference relation on $D_i$ defined by: for all $x_i, x_i' \in D_i$, $x_i \succ^{\mathbf{x}_i|\mathbf{x}_1=x_1,\ldots,\mathbf{x}_{i-1}=x_{i-1}} x_i'$ iff $x_1 \ldots x_{i-1} x_i x_{i+1} \ldots x_p \succ x_1 \ldots x_{i-1} x_i' x_{i+1} .. x_p$ holds for all $(x_{i+1}, \ldots, x_p) \in D_{i+1} \times \ldots \times D_p$.

Due to the fact that $\succ \in Legal(O)$ and that $\succ$ is a linear order, $\succ^{\mathbf{x}_i|\mathbf{x}_1=x_1,\ldots,\mathbf{x}_{i-1}=x_{i-1}}$ is a well-defined linear order as well. Note also that if $\succ$ is legal with respect to both $O = \mathbf{x}_1 > ... > \mathbf{x}_p$ and $O' = \mathbf{x}_{\sigma(1)} > ... > \mathbf{x}_{\sigma(k-1)} > \mathbf{x}_i (= \mathbf{x}_{\sigma(k)}) > ... > \mathbf{x}_{\sigma(p)}$, then $\succ^{\mathbf{x}_i|\mathbf{x}_1=x_1,\ldots,\mathbf{x}_{i-1}=x_{i-1}}$ and $\succ^{\mathbf{x}_i|\mathbf{x}_{\sigma(1)}=x_{\sigma(1)},\ldots,\mathbf{x}_{\sigma(k-1)}=x_{\sigma(k-1)}}$ coincide. In other words, the local preference relation on $\mathbf{x}_i$ depends only on the values of the variables that precede $\mathbf{x}_i$ in $O$ *and* in $O'$.

Lastly, for any acyclic graph $G$ over $I$, we say that two linear preference relations $V_1$ and $V_2$ are $G$-equivalent, denoted by $V_1 \sim_G V_2$, if and only if $V_1$ and $V_2$ are both compatible with $G$ and for any $\mathbf{x} \in I$, for any $\vec{y}, \vec{y}' \in D_{Par_G(\mathbf{x}))}$ we have $V_1^{\mathbf{x}|Par_G(\mathbf{x})=\vec{y}} = V_2^{\mathbf{x}|Par_G(\mathbf{x})=\vec{y}}$. The following observation is direct from this definition.

**Observation 1** *For any linear preference relations $V_1$ and $V_2$, $V_1 \sim_G V_2$ if and only if there exists a CP-net $\mathcal{N}$ whose associated graph is $G$ and such that $V_1$ and $V_2$ both extend $\succ_{\mathcal{N}}$.*

**Example 2** *Let $I = \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$, all three being binary. and let $V$ and $V'$ be the following linear preference relations:*

$$V : xyz \succ xy\bar{z} \succ x\bar{y}\bar{z} \succ x\bar{y}z \succ \bar{x}y\bar{z} \succ \bar{x}\bar{y}\bar{z} \succ \bar{x}yz \succ \bar{x}\bar{y}z$$
$$V' : xyz \succ xy\bar{z} \succ \bar{x}y\bar{z} \succ x\bar{y}\bar{z} \succ \bar{x}yz \succ \bar{x}\bar{y}\bar{z} \succ x\bar{y}z \succ \bar{x}\bar{y}z$$

*Let $G$ the graph over $I$ whose set of edges is $\{(\mathbf{x}, \mathbf{z}), (\mathbf{y}, \mathbf{z})\}$. $V$ and $V'$ are both compatible with $G$. Moreover, $V \sim_G V'$, since all local preference relations coin-*

*cide:* $x \succ^{\mathbf{x}}_{V} \bar{x}$ *and* $x \succ^{\mathbf{x}}_{V'} \bar{x}$*;* $z \succ^{\mathbf{z}|\mathbf{x}=x,\mathbf{y}=y}_{V} \bar{z}$ *and* $z \succ^{\mathbf{z}|\mathbf{x}=x,\mathbf{y}=y}_{V'} \bar{z}$*; etc. The CP-net* $\mathcal{N}$ *such that* $V$ *and* $V'$ *both extend* $\succ_{\mathcal{N}}$ *is defined by the following local conditional preferences:* $x \succ \bar{x}$*;* $y \succ \bar{y}$*;* $xy : z \succ \bar{z}$*;* $x\bar{y} : \bar{z} \succ z$*;* $\bar{x}y : \bar{z} \succ z$*;* $\bar{x}\bar{y} : \bar{z} \succ z$*. Note that* $V$ *and* $V'$ *are legal with respect to both* $\mathbf{x} > \mathbf{y} > \mathbf{z}$ *and* $\mathbf{y} > \mathbf{x} > \mathbf{z}$*.*

## 2.3 $G$-legal profiles

Let $I$ be a set of issues, with $|I| \geq 2$, and $\{1, \ldots, N\}$ a set of voters, with $N \geq 2$. We now define a crucial domain restriction for the rest of the paper:

**Definition 1** *Given an acyclic graph* $G$ *on* $I$*, we define* $Legal(G)$ *as the set of all collective profiles* $P = (V_1, \ldots, V_N)$ *such that each* $V_i$ *is compatible with* $G$*.*

The following observation is straightforward but important:

**Observation 2** $P \in Legal(G)$ *if and only if* $P \in Legal(O)$ *for all* $O$ *following* $G$*.*

We might wonder how strong the restriction to $O$-legal profiles is.

First, this restriction is much less demanding than separability. To see this, let $G_\emptyset$ the graph whose set of vertices is $I$ and that contains no edge; then we have the following important fact (whose proof is obvious):

**Observation 3** *The following three assertions are equivalent:*

1. $V \in Legal(G_\emptyset)$.
2. *for any order* $O$ *on* $I$*,* $V \in Legal(O)$.
3. $V$ *is separable.*

Thus, $O$-legality is a family of domain restrictions that includes separability as a special case but contains much more profiles[5].

Second, in many real-life domains it may be deemed reasonable to assume that preferential dependencies between variables coincide for all voters. For instance, in a designated-seat election process [4] where an assembly composed

[5] Notice that while there are only $2^p$ different CP-nets whose associated graph is $G_\emptyset$, there are $2^{2^p-1}$ CP-nets whose associated graph is the acyclic graph containing an edge $(x_i, x_j)$ for each $i < j$.

of a president, a vice-president and a secretary has to be elected, it may be intuitively reasonable to assume that voters' preferences are compatible with the order `president` $>$ `vice-president` $>$ `treasurer` .

Third, having $P_1 \in Legal(G_1)$ and $P_2 \in Legal(G_2)$ for $G_2 \neq G_1$ does not mean that a profile containing $P_1$ and $P_2$ is not $G$-legal for some acyclic graph $G$. Indeed, suppose that the linear preference relations $(\succ_1, \ldots, \succ_N)$ are compatible with the acyclic graphs $G_1, \ldots, G_N$, whose sets of edges are $E_1, \ldots, E_N$. Then they are a fortiori compatible with the graph $G^*$ whose set of edges is $E_1 \cup \ldots \cup E_N$. Therefore, *if $G^*$ is acyclic*, then sequential voting will be applicable to $(\succ_1, \ldots, \succ_N)$ (of course, this is no longer true if $G^*$ has cycles).

## 2.4 Sequential voting rules and correspondences

From now on, we assume that the set of candidates is a multi-issue domain $\mathcal{X} = D_1 \times \ldots \times D_p$. *Sequential voting* consists in applying "local" voting rules or correspondences on single issues, one after the other, in such an order that the local vote on a given issue can be performed only when the local votes on all its parents in the graph $G$ have been performed.

**Definition 2** *Let $G$ be an acyclic graph on $I$; let $P = (V_1, ..., V_N)$ in $Legal(G)^N$, $O = \mathbf{x}_1 > \ldots > \mathbf{x}_p$ a linear order on $V$ following $G$, and $(r_1, \ldots, r_p)$ a collection of deterministic voting rules (one for each variable $\mathbf{x}_i$). The* sequential voting rule *$Seq(r_1, \ldots, r_p)$ is defined as follows:*

- $x_1^* = r_1(V_1^{\mathbf{x}_1}, \ldots, V_N^{\mathbf{x}_1})$;
- $x_2^* = r_2(V_1^{\mathbf{x}_2 | \mathbf{x}_1 = x_1^*}, \ldots, V_N^{\mathbf{x}_2 | \mathbf{x}_1 = x_1^*})$;
- $\ldots$
- $x_p^* = r_p(V_1^{\mathbf{x}_p | \mathbf{x}_1 = x_1^*, .., \mathbf{x}_{p-1} = x_{p-1}^*}, .., V_N^{\mathbf{x}_p | \mathbf{x}_1 = x_1^*, .., \mathbf{x}_{p-1} = x_{p-1}^*})$

*Then $Seq(r_1, \ldots, r_p)(P) = (x_1^*, \ldots, x_p^*)$.*

**Example 3** *Let $N = 8$, $I = \{\mathbf{x}, \mathbf{y}\}$ with $D_\mathbf{x} = \{x_1, x_2, x_3\}$ and $D_\mathbf{y} = \{y, \bar{y}\}$, and $P = (V_1, \ldots, V_8)$ the following 8-voter profile:*

| | |
|---|---|
| $V_1, V_2, V_3$ : | $x_1\bar{y} \succ x_1 y \succ x_2\bar{y} \succ x_2 y \succ x_3 y \succ x_3\bar{y}$ |
| $V_4, V_5$: | $x_2 y \succ x_3 y \succ x_2\bar{y} \succ x_1 y \succ x_3\bar{y} \succ x_1\bar{y}$ |
| $V_6$: | $x_3\bar{y} \succ x_1\bar{y} \succ x_3 y \succ x_1 y \succ x_2 y \succ x_2\bar{y}$ |
| $V_7, V_8$: | $x_3\bar{y} \succ x_3 y \succ x_2 y \succ x_2\bar{y} \succ x_1 y \succ x_1\bar{y}$ |

*All these linear preference relations are compatible with the graph G over* $\{\mathbf{x}, \mathbf{y}\}$ *whose single edge is* $(\mathbf{x}, \mathbf{y})$*; equivalently, they follow the order* $\mathbf{x} > \mathbf{y}$*. Hence,* $P \in Legal(G)$*. The corresponding conditional preference tables are:*

| *voters 1,2,3* | *voters 4,5* | *voter 6* | *voters 7,8* |
|---|---|---|---|
| $x_1 \succ x_2 \succ x_3$ | $x_2 \succ x_3 \succ x_1$ | $x_3 \succ x_1 \succ x_2$ | $x_3 \succ x_2 \succ x_1$ |
| $x_1 : \bar{y} \succ y$ | $x_1 : y \succ \bar{y}$ | $x_1 : \bar{y} \succ y$ | $x_1 : y \succ \bar{y}$ |
| $x_2 : \bar{y} \succ y$ | $x_2 : y \succ \bar{y}$ | $x_2 : y \succ \bar{y}$ | $x_2 : y \succ \bar{y}$ |
| $x_3 : y \succ \bar{y}$ | $x_3 : \bar{y} \succ y$ | $x_3 : \bar{y} \succ y$ | $x_3 : \bar{y} \succ y$ |

*Take first* $r_{\mathbf{x}}$ *equal to the Borda rule, plus a tie-breaking mechanism that favors* $x_1$ *over* $x_2$ *and* $x_3$*, and* $x_2$ *over* $x_3$*, and* $r_{\mathbf{y}}$ *to the majority rules, plus a tie-breaking mechanism that favors* $y$ *over* $\bar{y}$*. The projection of P on* $\mathbf{x}$*, namely* $P^{\mathbf{x}} = (V_1^{\mathbf{x}}, \ldots, V_8^{\mathbf{x}})$*, contains three votes* $x_1 \succ x_2 \succ x_3$*, two votes* $x_2 \succ x_3 \succ x_1$*, one vote* $x_3 \succ x_1 \succ x_2$ *and two votes* $x_3 \succ x_2 \succ x_1$*, therefore, the Borda winner for* $P^{\mathbf{x}}$ *is* $x^* = r_{\mathbf{x}}(P^{\mathbf{x}}) = x_2$*. Now, the projection of P on* $\mathbf{y}$ *given* $\mathbf{x} = x_2$*, namely* $P^{\mathbf{y}|\mathbf{x}=x_2} = (V_1^{\mathbf{y}|\mathbf{x}=x_2}, \ldots, V_8^{\mathbf{y}|\mathbf{x}=x_2})$*, is composed of 5 votes for y and 3 for* $\bar{y}$*, therefore* $y^* = r_{\mathbf{y}}(P^{\mathbf{y}|\mathbf{x}=x_2}) = y$*. The sequential winner is now obtained by combining the* $\mathbf{x}$*–winner and the conditional* $\mathbf{y}$*-winner given* $\mathbf{x} = x^*$*, namely* $Seq_{r_{\mathbf{x}}, r_{\mathbf{y}}}(P) = x_2 y$*.*

*If we take* $r_{\mathbf{x}}$ *equal to the plurality rule instead of the Borda rule (plus the same tie-breaking mechanisms), then we get* $x^* = r_{\mathbf{x}}(P^{\mathbf{x}}) = x_1$*, and* $y^* = r_{\mathbf{y}}(P^{\mathbf{y}|\mathbf{x}=x_1}) = y$*, therefore* $Seq_{r_{\mathbf{x}}, r_{\mathbf{y}}}(P) = x_1 y$*.*

In addition to sequential voting rules, we also define *sequential voting correspondences* in a similar way: if for each $i$, $C_i$ is a correspondence on $D_i$, then $Seq(C_1, \ldots, C_p)(P)$ is the set of all outcomes $(x_1, \ldots, x_p)$ such that $x_1 \in C_1(V_1^{\mathbf{x}_1}, \ldots, V_N^{\mathbf{x}_1})$, and for all $i \geq 2$, $x_i \in C_i(V_1^{\mathbf{x}_i|\mathbf{x}_1=x_1,..,\mathbf{x}_{i-1}=x_{i-1}}, .., V_N^{\mathbf{x}_i|\mathbf{x}_1=x_1,..,\mathbf{x}_{i-1}=x_{i-1}})$.

For instance, in Example 3, if we take $C_{\mathbf{x}}$ to be the plurality correspondence and $C_{\mathbf{y}}$ to be the majority correspondence, then $C_{\mathbf{x}}(P^{\mathbf{x}}) = \{x_1, x_3\}$, and $C_{\mathbf{y}}(P^{\mathbf{y}|\mathbf{x}=x_1}) \cup C_{\mathbf{y}}(P^{\mathbf{y}|\mathbf{x}=x_3}) = \{x_1 y, x_1\bar{y}\} \cup \{x_3 y\} = \{x_1 y, x_1\bar{y}, x_3 y\}$.

## 2.5 Seat-by-seat voting rules

Seat-by-seat voting rules are the sequential voting rules whose domain is the set of all separable profiles. By Observation 3 the set of all separable profiles is denoted by $Legal(G_\emptyset)$ or equivalently $\bigcap_O Legal(O)$. So every separable $V$ can be projected to each local domain $D_i$ as $V^{\mathbf{x}_i}$ or equivalently $V|_{\mathbf{x}_i}$ without conditioning on the valuation of other issues. Denote $SbS(c_1,\ldots,c_p)$ the seat-by-seat correspondence of $c_1,\ldots,c_p$. It is indeed the restriction of $Seq(c_1,\ldots,c_p)$ on $Legal(G_\emptyset)$, namely for any profile $P$

$$SbS(c_1,\ldots,c_p)(P) = \begin{cases} Seq(c_1,\ldots,c_p)(P) & \text{if} \quad P \in Legal(G_\emptyset) \\ \text{Undefined} & \text{otherwise} \end{cases}$$

There are two important theorems proved in [4] about the neutrality and efficiency of seat-by-seat voting rules. First we recall the definition of neutrality and efficiency for decomposable rules.

**Definition 1** *A sequential voting rule $Seq(r_1,\ldots,r_p)$ is* neutral *if for any permutation $M$ on $\mathcal{X}$ and any profile $P \in Legal(O)$ where $O = \mathbf{x}_1 > \ldots \mathbf{x}_p$, whenever $M(P) \in Legal(O)$,*

$$M(Seq(r_1,\ldots,r_p)(P)) = Seq(r_1,\ldots,r_p)(M(P)).$$

The neutrality of sequential voting correspondence is defined similarly.

**Definition 2** *A sequential voting rule $Seq(r_1,\ldots,r_p)$ is* efficient *if for any profile $P \in Legal(O)$, whenever there exist $\alpha,\beta \in \mathcal{X}$ such that $\alpha \succ_V \beta$ for all $V \in P$, then $Seq(r_1,\ldots,r_p)(P) \neq \beta$.*

We say a sequential voting correspondence $Seq(c_1,\ldots,c_p)$ is efficient if whenever $\alpha \succ_V \beta$ for all $V \in P$, we have:

$$\beta \in Seq(c_1,\ldots,c_p)(P) \text{ implies } \alpha \in Seq(c_1,\ldots,c_p)(P).$$

We also need the notion of *local efficiency*.

**Definition 3** *A sequential voting rule $Seq(r_1,\ldots,r_p)$ is* locally efficient *if for every $i \leq n$, $r_i$ is efficient.*

For seat-by-seat votes and correspondences, we just replace $Legal(O)$ by $Legal(G_\emptyset)$ in Definition 1 and Definition 2.

In the remainder of the paper, we will always assume $D_i = \{0_i, 1_i, \ldots, a_i\}$.

We now recall these two theorems from [4].

**Theorem 1 (Theorem 1 in [4])** *When the domain is not composed of two binary issues, then the only efficient voting rule is a dictatorship.*

**Theorem 2 (Theorem 2 in [4])** *Suppose $p \geq 3$ and $r_i$ satisfies Pareto optimality for all $i \leq p$. If $SbS(r_1, \ldots, r_p)$ is neutral, then it must be a dictatorship.*

# 3 Impossibility theorems

In this section we prove two impossibility theorems about the existence of neutral or efficient seat-by-seat correspondences. The proof can be easily extended to voting rules. Then we show that such results also hold for decomposable voting rules and correspondences, thus completely answering the question of whether there exist neutral or efficient decomposable voting rules or correspondences.

**Definition 4** *A rule or correspondence is a* dictatorship *for voter i, if*

$$\forall P, r(P) = (V_i)_1.$$

*It is an* antidictatorship *, if*

$$\forall P, r(P) = (V_i)_{|D_i|}.$$

*The* trivial *correspondence is the correspondence C that outputs X for any N-voter profile, i.e., $C(P) = X$ for every P.*

We first show that if one of the local correspondences $x_i$ is dictatorial and $SbS(c_1, \ldots, c_p)$ is neutral or efficient then $SbS(c_1, \ldots, c_p)$ is dictatorial.

**Theorem 3** *If $SbS(c_1, \ldots, c_p)$ satisfies neutrality or efficiency, and is not a dictatorship, then for all $i \leq p$, $c_i$ is not a dictatorship.*

Then we show that the local efficiency condition in Theorem 2 can be removed, with the conclusion being slightly changed, because the next theorem tells us neutrality of seat-by-seat voting rule will induce local efficiency or local anti-efficiency.

**Theorem 4** *If* $SbS(c_1,\ldots,c_p)$ *is neutral, then one of the following two conditions holds:*

1. *for all* $i \leq p$, $c_i$ *satisfies efficiency.*

2. *for all* $i \leq p$, $c_i$ *satisfies anti-efficiency.*

The next theorem says when $p \geq 3$, if a decomposable correspondence satisfies neutrality or efficiency, then it must be either dictatorial or trivial.

**Theorem 5** *Let* $p \geq 3$, $N \in \mathbb{N}$, *and let* $SbS(c_1,\ldots,c_p)$ *be a sequential rule over* $N$*-voter separable profiles. Then we have the following:*

1. *If* $SbS(c_1,\ldots,c_p)$ *satisfies neutrality then it is either a dictatorship, an anti-dictatorship, or the trivial correspondence.*

2. *If* $SbS(c_1,\ldots,c_p)$ *satisfies efficiency then it is either a dictatorship or the trivial correspondence.*

As a corollary of point 1 above, if $SbS(c_1,\ldots,c_p)$ satisfies either neutrality and local efficiency then it is either a dictatorship or the trivial correspondence.

Lastly we consider $p = 2$ case.

**Theorem 6** *Suppose* $p = 2$ *and at least one of* $|D_1|$ *and* $|D_2|$ *is three or more, then the only seat-by-seat correspondence satisfying neutrality or efficiency on* $N$*-voters is a (anti-)dictatorship, or the trivial one.*

Notice that our proofs is directly applied to seat-by-seat rules. So combining Theorem 5, Theorem 6, and Theorem 1 and 2 in [4], we have the following theorem dealing with the neutrality or efficiency of seat-by-seat voting rule or correspondence.

**Theorem 7** *The only neutral or efficient seat-by-seat voting rule or correspondence on a combinatorial domain other than two-binary-issue is a (anti-)dictatorship or the trivial one.*

Because seat-by-seat rule (correspondence) is a decomposable voting rule (correspondence) restricted on a smaller domain (all separable profiles), so we immediately know the neutrality and efficiency of decomposable voting rule (correspondence) in the following corollary.

**Corollary 8** *The only neutral or efficient decomposable voting rule or correspondence on a combinatorial domain other than two-binary-issue is a dictatorship, a antidictatorship (only for neutrality case) or the trivial one.*

For the two-binary-issue combinatorial domain $\mathcal{X} = \{0_1, 1_1\} \times \{0_2, 1_2\}$, it has already been proved that the sequential majority correspondence is neutral and efficient [13]. So the existence of neutral or efficient decomposable correspondence is clear.

# 4 Conclusion

In this paper we focus on the existence of neutral or efficient seat-by-seat voting correspondence, and show such a correspondence must be a dictatorship or the trivial one. Our results refine the theorems by Benoit and Kornhauser, and solved the remaining uncovered cases. The results also give an affirmative answer to the conjecture of J. Lang on the neutrality of decomposable voting correspondences.


# Acknowledgement

The author wants to thank Patrick Girard and Yoav Shoham for helpful discussions and comments.


# References


[1] J.-P. Benoit and L Kornhauser. Social choice in a representative democracy. *American Political Science Review*, 88(1):185–192, 1994.

[2] J.-P. Benoit and L Kornhauser. On the assumption of separable assembly preferences. *Social Choice and Welfare*, 16(3):429–439, 1999.

[3] J.-P. Benoit and L.A. Kornhauser. Voting simply in the election of assemblies. Technical Report RR# 91-32, C.V. Starr Center Working Papers, 1991.

[4] J.-P. Benoit and L.A. Kornhauser. Only a dictatorship is efficient or neutral. Technical report, 2006.

[5] C. Boutilier, R. Brafman, C. Domshlak, H. Hoos, and D. Poole. Cp-nets: a tool for representing and reasoning with conditional *ceteris paribus* statements. *Journal of Artificial Intelligence Research*, 21:135–191, 2004.

[6] C. Boutilier, R. I. Brafman, C. Domshlak, H. Hoos, and D. Poole. Preference-based constrained optimization with CP-nets. *Computational Intelligence*, 20(2):137–157, 2004.

[7] S. J. Brams, D. M. Kilgour, and W. S. Zwicker. Voting on referenda: the separability problem and possible solutions. *Electoral Studies*, 16(3):359–377, 1997.

[8] J. Goldsmith, J. Lang, M. Truszczinski, and N. Wilson. The computational complexity of dominance and consistency in CP-nets. In *Proceedings of the Nineteenth International Joint Conference on Artificial intelligence (IJCAI-05*, pages 144–149, 2005.

[9] J. Hodge. *Separable preference orders*. PhD thesis, Western Michigan University, 2002.

[10] R. Keeney and H. Raiffa. *Decision with Multiple Objectives: Preferences and Value Tradeoffs*. Wiley and Sons, 1976.

[11] D. Lacy and E. Liou. A problem with referenda. *Journal of Theoretical Politics*, 12(1):5–31, 2000.

[12] J. Lang. Voting and aggregation on combinatorial domains with structured preferences. In *Proceedings of the Twenthieth Joint International Conference on Artificial Intelligence (IJCAI'07)*, pages 1366–1371, 2007.

[13] L. Xia, J. Lang, and M. Ying. Sequential voting and multiple election paradoxes. In *Proceedings of the 11th Conference on the Theoretical Aspects of Rationality and Knowledge (TARK 2007)*, 2007.

[14] L. Xia, J. Lang, and M. Ying. Strongly decomposable voting rules on multiattribute domains. In *Proceedings of the 22nd National Conference on Artificial Intelligence (AAAI 2007)*, 2007.

## Lemmas and proofs

The following lemma will be frequently used in our proofs.

**Lemma 9** *Given a CP-net $\mathcal{N}$, if $\mathcal{N} \not\models \beta \succ \alpha$ and $\mathcal{N} \not\models \alpha \succ \beta$, there exists a linear preference $V$ extending $\mathcal{N}$ such that*

1. $\alpha \succ_V \beta$.
2. *$\alpha$ and $\beta$ are adjacent, i.e., there is no $\gamma$ such that $\alpha \succ_V \gamma \succ_V \beta$.*
3. *Let $V'$ be obtained from $V$ by exchanging $\alpha$ and $\beta$, then $V' \sim_{\mathcal{N}} V$.*

**Proof of Lemma 9:** To prove this lemma, we first "merge" $\alpha$ and $\beta$ by removing them from $\mathcal{X}$, and replace them by a new member $\theta$. Define $\mathcal{X}' = \mathcal{X} \cup \{\theta\} - \{\alpha, \beta\}$ and a preorder $\succ'$ on $\mathcal{X}'$ by $\vec{x} \succ' \vec{y}$ iff one of the following conditions holds:

1. $\vec{x} = \theta$, and $\alpha \succ_{\mathcal{N}} \vec{y}$ or $\beta \succ_{\mathcal{N}} \vec{y}$,
2. $\vec{y} = \theta$, and $\vec{x} \succ_{\mathcal{N}} \alpha$ or $\vec{x} \succ_{\mathcal{N}} \beta$.
3. $\vec{x} \succ_{\mathcal{N}} \vec{y}$.

To see $\succ'$ is well-defined, we only need to check that

a. if $\vec{x} \succ_{\mathcal{N}} \alpha$ and $\beta \succ_{\mathcal{N}} \vec{y}$ then $\mathcal{N} \not\models \vec{y} \succ \vec{x}$, and

b. if $\vec{x} \succ_{\mathcal{N}} \beta$ and $\alpha \succ_{\mathcal{N}} \vec{y}$ then $\mathcal{N} \not\models \vec{y} \succ \vec{x}$.

(a) is because if $\vec{y} \succ_{\mathcal{N}} \vec{x}$, then from transitivity $\beta \succ_{\mathcal{N}} \alpha$, contradicting the assumption. Similarly (b) is true. So $\succ'$ is well-defined.

Choose any linear preorder $V^*$ on $\mathcal{X}'$ that extends $\succ'$. Then we expand $V^*$ to $V$ by removing $\theta$ and adding $\alpha \succ_V \beta$, and let $\alpha, \beta$ inherit the position of $\theta$. Formally, $\alpha \succ_V \vec{y}$ iff $\theta \succ_{V^*} \vec{y}$ and $\vec{x} \succ_V \alpha$ iff $\vec{x} \succ_{V^*} \theta$, and similar preferences for $\beta$. It is easy to see that $V$ extends $\mathcal{N}$, and satisfies 1. and 2. If we added $\beta \succ_V \alpha$ instead of $\alpha \succ_V \beta$, then we obtain $V'$. So 3. also holds. □

Next some definitions and notations.

**Definition 5** *Two candidates $\vec{d}_1, \vec{d}_2 \in D_1 \times \ldots \times D_p$ are said to be exchangeable in a separable vote $V$, iff*

1. *$\vec{d}_1$ and $\vec{d}_2$ are adjacent in $V$, and*

2. *let $M$ be the permutation than only exchanges $\vec{d}_1$ and $\vec{d}_2$, then for all $i \leq p$, $V|_{D_i} = M(V|_{D_i})$.*

$\vec{d}_1$ and $\vec{d}_2$ are exchangeable in a vote $V$ means that after exchanging them, the local preference on any $D_i$ does not change.

**Definition 6** *Given a set of separable CP-nets $\mathfrak{N} = (\mathcal{N}_1, \ldots, \mathcal{N}_N)$, define $\mathcal{N}_j|_{D_i}$ be the projection of $\mathcal{N}_j$ on $D_i$, we also write $\mathfrak{N}|_{D_i} = (\mathcal{N}_1|_{D_i}, \ldots, \mathcal{N}_N|_{D_i})$.*

**Definition 7** *For all $k \leq N$, define $R_i^k = (V_1^i, \ldots, V_N^i)$ be the set of profiles on $D_i$ s.t.*

$$V_j^i = \begin{cases} 0_i \succ \ldots \succ a_i & \text{if } j \leq k \\ a_i \succ \ldots \succ 0_i & \text{if } j > k \end{cases}$$

**Definition 8** *We say the relative preference between $\alpha$ and $\beta$ is not determined in check if this can be removed. a linear order $V$ extending $\mathcal{N}$, if $\mathcal{N} \not\models \beta \succ \alpha$ and $\mathcal{N} \not\models \alpha \succ \beta$. In this case we also say that the relative preference between $\alpha$ and $\beta$ is not determined in $\mathcal{N}$.*

The next Lemma provides an easy way to check whether the relative preference between $\alpha$ and $\beta$ is determined in a CP-net with no edge.

**Lemma 10** *Suppose $\mathcal{N}$ is a CP-net whose graph does not have any edge, then the relative preference between $\alpha$ and $\beta$ is not determined in $\mathcal{N}$ if and only if*

$$\text{there exist } i, j \leq p \text{ such that } \mathcal{N}|_{D_i} \models \alpha_i \succ \beta_i \text{ and } \mathcal{N}|_{D_j} \models \beta_j \succ \alpha_j$$

[6]

**Lemma 11** *Given $N$ CP-nets $\mathcal{N}_1, \ldots, \mathcal{N}_N$, if for all $j \leq N$, the relative preference between $\alpha$ and $\beta$ is not determined in $\mathcal{N}$, then for any $SbS(c_1, \ldots, c_p)$ satisfying neutrality or efficiency, and for any profile $P$ extending the CP-nets $\mathcal{N}_1, \ldots, \mathcal{N}_N$,*

$$\beta \in SbS(c_1, \ldots, c_p)(P) \Longleftrightarrow \alpha \in SbS(c_1, \ldots, c_p)(P).$$

[6] Jérôme : *I agree there's no need to give a proof of this lemma, as it is a well-known result for separable preferebce – I'll try to find the reference for this result*

**Proof of Lemma 11:** Without loss of generality, we only prove the $\Rightarrow$ part. Now suppose the lemma does not hold, then there exists $c_1 \dots, c_p$ and a separable profile $P' = (V'_1, \dots, V'_M)$ s.t. $SbS(c_1, \dots, c_p)$ is neutral or efficient, and $\beta \in SbS(c_1, \dots, c_p)(P')$ but $\alpha \notin SbS(c_1, \dots, c_p)(P')$.

Denote $\mathcal{N}_1, \mathcal{N}_N$ the CP-nets $P'$ extends. Since the relative preference between $\alpha$ and $\beta$ is not determined in any $\mathcal{N}_j$, then by Lemma 10, there exists $P = (V_1, \dots, V_N)$ extending $\mathcal{N}_1, \dots, \mathcal{N}_N$ respectively, and satisfies

1. $\alpha \succ_{V_j} \beta$ in all $j \leq N$,
2. $V_j \sim_{\mathcal{N}_j} M(V_j)$, where $M$ is the permutation that only exchanges $\alpha$ and $\beta$.

By definition

$$SbS(c_1, \dots, c_p)(P') = SbS(c_1, \dots, c_p)(P).$$

If $SbS(c_1, \dots, c_p)$ satisfies efficiency, then from condition 1. we know $\alpha \in SbS(c_1, \dots, c_p)(P) = SbS(c_1, \dots, c_p)(P')$, contradiction. If $SbS(c_1, \dots, c_p)$ satisfies neutrality, then

$$\begin{aligned}\alpha =& M(\beta) \in M(SbS(c_1, \dots, c_p)(P)) = SbS(c_1, \dots, c_p)(M(P)) \\ =& SbS(c_1, \dots, c_p)(P) = SbS(c_1, \dots, c_p)(P')\end{aligned}$$

Contradiction again. So the lemma is true. □

**Lemma 12** *Let $c_1, \dots, c_p$ be correspondences. Assume $SbS(c_1, \dots, c_p)$ satisfies neutrality or efficiency, and that there exists $i \leq p$ such that one of the following two conditions hold:*

1. *there exists $d_i \in c_i(R_i^N)$ such that $d_i \notin \{0_i, a_i\}$*
2. *$\{0_i, a_i\} \subseteq c_i(R_i^N)$*

*Then $c_i(R_i^N) = D_i$ for all $i \leq p$.*

**Proof of Lemma 12:** Without loss of generality we assume $i = 1$. First we claim for any $k \neq 1$, $c_k(R_k^N) = D_k$. If not, then there exists $k \leq p$ and $e_k \in D_k$ such that $e_k \notin c_k(R_k^N)$. Without loss of generality, suppose $k = 2$, $r = 2$, and $d_2 \in c_2(R_2^N)$, $e_2 \notin c_2(R_2^N)$. We will prove the claim under the two different assumptions given in the statement of the lemma:

1. Assume $d_1 \notin \{0_1, a_1\}$. If $e_2 \succ d_2$, then in the CP-net $\mathcal{N} = (0_1 \succ \ldots \succ a_1, 0_2 \succ \ldots \succ a_2)$, neither $\mathcal{N} \not\models (d_1, d_2) \succ (a_1, e_2)$ nor $\mathcal{N} \not\models (a_1, e_2) \succ (d_1, d_2)$. Consider the CP-nets

$$\mathfrak{N} = (\mathcal{N}, \ldots, \mathcal{N})$$

Notice that $(d_1, d_2) \in SbS(c_1, \ldots, c_p)(R_1^N, R_2^N)$, by Lemma 11 we know $(a_1, e_2) \in SbS(c_1, \ldots, c_p)(R_1^N, R_2^N)$, which means $e_2 \in c_2(R_2^N)$, contradiction. If $e_2 \prec d_2$, we take the same CP-net $\mathcal{N}$, and consider the pair $(d_1, d_2)$ and $(0_1, e_2)$. Similarly there is a contradiction.

2. Assume $\{0_1, a_1\} \subseteq c_1(R_1^N)$. If $e_2 \succ d_2$ we consider the pair $(0_1, d_2), (a_1, e_2)$ in $\mathcal{N}$; if $e_2 \prec d_2$ we consider the pair $(a_1, d_2), (0_1, e_2)$. There is a similar contradiction with that in the first assumption, so $c_2(R_2^N) = D_2$.

Lastly, we still have to consider the case $k = 1$. Notice that $\{0_2, a_2\} \subseteq c_2(R_2^c)$. Thus, applying the above claim on $i = 2$, we get $c_1(R_1^N) = D_1$.

Therefore if one of the two assumptions holds, then the only seat-by-seat correspondence satisfying neutrality or efficiency is the trivial correspondence that always choose the whole candidate set. □

The only cases that are not covered by assumptions 1 and 2 of Lemma 12 are those where for every $i \leq p$, $c_i(R_i^N) = \{0_i\}$ or $c_i(R_i^N) = \{a_i\}$, that is, every $c_i$ outputs whether the best alternative or the worst alternative for all the voters. The next lemma considers the case where at least one of the $R_i$'s outputs the best alternative and at least one of the $c_i$'s outputs the worst alternative.

**Lemma 13** *Assume $SbS(c_1, \ldots, c_p)$ satisfies neutrality or efficiency, and that there exist $i$ and $j$ such that $0_i \in c_i(R_i^N)$ and $a_j \in c_j(R_j^N)$. Then $c_i(R_i^N) = D_i$ for all $i \leq p$.*

**Proof of Lemma 13:** Consider the pair $(0_i, a_j), (a_i, 0_j)$ then following a similar proof of Lemma 12 we know there $a_i \in c_i(R_i^N)$, $0_j \in c_j(R_j^N)$. So by Lemma 12, $c_i(R_i^N) = D_i$ for all $i \leq p$. □

From Lemmas 12 and 13 we know that if $SbS(c_1, \ldots, c_p)$ is neutral or efficient, then it must be one of the following three cases.

1. $\forall i \leq p, c_i(R_i^N) = D_i$

2. $\forall i \leq p, c_i(R_i^N) = \{0_i\}$

3. $\forall i \leq p, c_i(R_i^N) = \{a_i\}$

Now, from Theorem 4.1 in [13] we know that if $SbS(c_1,\ldots,c_p)$ is efficient, then each $c_i$ is efficient, that is, $SbS(c_1,\ldots,c_p)$ is locally efficient. Obviously, the local correspondence such that $c_i(R_i^N)=\{a_i\}$ is not efficient. Therefore, if $SbS(c_1,\ldots,c_p)$ is efficient, then case 3 is excluded.

**Lemma 14** *For any $SbS(c_1,\ldots,c_p)$ that is neutral or efficient, for any $i,j\leq p$,*

1. $0_i\in c_i(R_i^N)\Longleftrightarrow a_j\in c_j(R_j^0)$.
2. $c_i(R_i^N)=\{0_i\}\Longleftrightarrow c_j(R_j^0)=\{a_j\}$.

**Proof of Lemma 14:** The proof is very similar with that of Lemma 12. We only prove "⇒" of both parts, and we assume $i=1, j=2$. If this case can be proved, so can $i=j$ case.

Consider the CP-net $\mathcal{N}$ defined by

$$\begin{aligned}\mathcal{N}|_{D_1} &= 0_1\succ\ldots\succ a_1\\ \mathcal{N}|_{D_2} &= a_2\succ\ldots\succ 0_2\\ \mathcal{N}|_{D_j} &= 0_j\succ\ldots\succ a_j \text{ for all } j\geq 2\end{aligned}$$

Assume now $0_1\in c_1(R_1^N)$. We want to show that $a_2\in c_2(R_2^0)$.

For any $d_2\in D_2$, $d_2\neq a_2$, from Lemma 10 we know the relative preference between $(0_1,d_2)$ and $(a_1,a_2)$ is not determined, so for any $N$-voter profile $P$ s.t. each vote extends $\mathcal{N}$, if $d_2\in c_2(R_2^0)$ then

$$(0_1,d_2,0_3,\ldots,0_p)\in SbS(c_1,\ldots,c_p)(P)$$

By Lemma 11 we have

$$(a_1,a_2,0_3,\ldots,0_p)\in SbS(c_1,\ldots,c_p)(P)$$

which means that if for some $d_2\neq a_2$ we have $d_2\in c_2(R_2^0)$ then we have $a_2\in c_2(R_2^0)$, and vice versa.

To prove 1., choose any $d_2\in c_2(R_2^0)$, $d_2\neq a_2$, and from the argument above we know $a_2\in c_2(R_2^0)$, which means that $a_2$ is always in $c_2(R_2^0)$. This proves the ⇒ direction of 1. (The proof of the ⇐ direction is very similar.)

To prove 2., from $a_2\in c_2(R_2^0)$ we know for any $d_2\in D_2$, $d_2\in c_2(R_2^0)$, which means $c_2(R_2^0)=D_2$. □

**Proof of Theorem 3:** If the theorem does not hold, there exists $i$ such that $c_i$ is a dictatorship but $SbS(c_1,\dots,c_p)$ is not. Without loss of generality let $i=1$, and

$$c_1(V_1^1,\dots,V_N^1)=\{top(V_1^1)\}$$

for all $N$-voter profile on $D_1$. We show for any $j\le p$, $c_j$ is a dictatorship for the first voter $V_1$.

Without loss of generality we prove the theorem for $j=2$. If there is a profile $P^2=(V_1^2,\dots,V_N^2)$ on $D_2$ such that $c_2(P^2)\neq\{top(V_1^2)\}$, then we choose any $d_2\in c_2(P^2)$, and denote $e_2=top(V_1^2)$, and consider the CP-nets $\mathfrak{N}=(\mathcal{N}_1,\dots,\mathcal{N}_N)$ s.t. for all $j\le N$

1. for all $i\ge 3$, $\mathcal{N}_j|_{D_i}=0_i\succ\dots\succ a_i$,
2. $\mathcal{N}_j|_{D_1}=\begin{cases}0_1\succ\dots\succ a_1 & \text{if } e_2\succ_{V_j^2} d_2\\ a_1\succ\dots\succ 0_1 & \text{if } e_2\prec_{V_j^2} d_2\end{cases}$
3. $\mathcal{N}_j|_{D_2}=V_j^2$.

Since $c_1$ is a dictatorship for $V_1$, and $top(\mathcal{N}_1|_{D_1})=0_1$, for any profile $P$ extending $\mathfrak{N}$,

$$(0_1,d_2,0_3,\dots,0_p)\in SbS(c_1,\dots,c_p)(P).$$

Notice the relative preference between $(0_1,d_2,0_3,\dots,0_p)$ and $(a_1,e_2,0_3,\dots,0_p)$ is not determined, from Lemma 11 we know $(a_1,e_2,0_3,\dots,0_p)\in SbS(c_1,\dots,c_p)(P)$, which means $a_1\in c_1(P|_{D_1})$, contradiction to the dictatorship of $c_1$ that $c_1(P|_{D_1})=\{0_1\}$.

So for any $i\le p$, $c_i$ is a dictatorship for voter 1, which means $SbS(c_1,\dots,c_p)$ is a dictatorship for voter 1, contradiction to the assumption that $SbS(c_1,\dots,c_p)$ is not a dictatorship. The proof is complete. □

**Lemma 15** *Suppose one of the following holds*

1. $p\ge 3$, *or*
2. $p=2$ *and at least one of* $|D_1|$ *and* $|D_2|$ *is no less than 3.*

*Assume furthermore that* $SbS(c_1,\dots,c_p)$ *is efficient. If there exists an N-vote profile P such that* $SbS(c_1,\dots,c_p)(P)\neq X$*, then* $c_i(R_i^N)=\{0_i\}$ *and* $c_i(R_i^0)=\{a_i\}$ *for all* $i\le p$.

**Proof of Lemma 15:** In fact, under all the assumptions of this lemma, $c_i(R_i^N) = \{0_i\}$ and $c_i(R_i^0) = \{a_i\}$ are equivalent by Lemma 14. So we fucus on proving $c_i(R_i^0) = \{a_i\}$.

Assume the Lemma does not hold, that is: (a) either $p \geq 3$, or $p = 2$ and (without loss of generality) $|D_1| \geq 3$; (b) $SbS(c_1, \dots, c_p)$ is efficient; (c) there exists an N-vote profile $P$ s.t. $SbS(c_1, \dots, c_p)(P) \neq \mathcal{X}$; and (d) there exists $l \leq p$ s.t.$c_l(R_l^N) \neq \{0_l\}$.

(d) implies that either $c_l(R_l^N)$ contains some $d_l \neq 0_l, a_l$, or $c_l(R_l^N) = \{0_l, a_l\}$, or $c_l(R_l^N) = \{a_l\}$. The latter case is impossible because $SbS(c_1, \dots, c_p)$ is efficient, which implies that $c_k$ is efficient [13], and, by Lemma 12, the first two cases, together with the efficiency of $SbS(c_1, \dots, c_p)$, imply that

(e) for any $j \leq p$, $c_j(R_j^N) = D_j$.

(c) implies that there exists a profile $P^i = (V_1^i, \dots, V_N^i)$ on $D_i$ such that $c_i(P^i) \neq D_i$. Without loss of generality, we assume $i = 1$, i.e.,

(f) $c(P^1) \neq D_1$

Choose any $d_1 \in c_1(P^1)$ and any $e_1 \notin c_1(P^1)$, consider a profile $Q^2 = (Q_1^2, \dots, Q_N^2)$ on $D_2$ such that

$$Q_j^2 = \begin{cases} 0_2 \succ \dots \succ a_2 & \text{iff } d_1 \succ_{V_j^1} e_1 \\ a_2 \succ \dots \succ 0_2 & \text{iff } e_1 \succ_{V_j^1} d_1 \end{cases}$$

We claim

**Claim 1** $c_2(Q^2) = D_2$.

**Proof of Claim 1:** First we consider the case $p \geq 3$. If $c_2(Q^2) \neq D_2$, then choose any $d_2 \in c_2(Q^2)$ and $e_2 \notin c_2(Q^2)$. We consider a set of CP-nets $\mathfrak{N} = (\mathcal{N}_1, \dots, \mathcal{N}_N)$ s.t. for all $j \leq N$,

1. $\mathcal{N}_j|_{D_2} = Q_j^2$;
2. for all $i \neq 2$, $\mathcal{N}_j|_{D_i} = 0_i \dots \succ a_i$;

Notice that $0_1 \succ_{\mathcal{N}_j|_{D_1}} a_1$ and $0_3 \succ_{\mathcal{N}_j|_{D_3}} a_3$. By Lemma 10, the preference between $(0_1, d_2, a_3, 0_4, \dots, 0_p)$ and $(a_1, e_2, 0_3, 0_4, \dots, 0_p)$ is not determined. Notice that $\mathcal{N}|_{D_1} = (0_1 \succ \dots \succ a_1, \dots, 0_1 \succ \dots \succ a_1) = R_1^N$, and by (e), $c_i(R_i^N) = D_i$ for all $i \leq p$, therefore $0_1 \in c_1(\mathcal{N}|_{D_1})$. Similarly, $a_3 \in c_3(\mathcal{N}|_{D_3})$. Because $d_2 \in c_2(Q^2)$, we have that $(0_1, d_2, a_3, 0_4, \dots, 0_p) \in SbS(c_1, \dots, c_p)(P)$ for any $P$ extending $\mathfrak{N}$. Then, by Lemma 11, $(a_1, e_2, 0_3, 0_4, \dots, 0_p) \in SbS(c_1, \dots, c_p)(P)$. So $e_2 \in c_2(Q^2)$, a contradiction. Therefore when $p \geq 3$, $c_2(Q^2) = D_2$.

Now we consider the case where $p = 2$ and (without loss of generality) $|D_1| > 2$. We want to show that in this case as well we have $c_2(Q^2) = D_2$. Consider the profile $Q^1 = (Q^1_1, \ldots, Q^1_N)$ on $D_1$ s.t. for all $j \leq N$

$$Q^1_j = \begin{cases} 0_1 \succ 1_1 \succ 2_1 \succ 3_1 \succ \ldots \succ a_1 & \text{if } Q^2_j = 0_2 \succ \ldots \succ a_2 \\ 0_1 \succ 2_1 \succ 1_1 \succ 3_1 \succ \ldots \succ a_1 & \text{if } Q^2_j = a_2 \succ \ldots \succ 0_2 \end{cases}$$

We claim $c_1(Q^1) = D_1$. To see this, consider CP-nets $\mathfrak{N} = (\mathcal{N}_1, \ldots, \mathcal{N}_N)$ s.t. for any $j \leq N$,

$$\mathcal{N}_j = (Q^1_j, 0_2 \succ \ldots \succ a_2).$$

For any $d'_1 \in D_1$ and $d'_1 \neq 0_1$, notice $\mathcal{N}_j \not\models (0_1, a_2) \succ (d'_1, 0_2)$, $\mathcal{N}_j \not\models (d'_1, 0_2) \succ (0_1, a_2)$, and $c_2(\mathfrak{N}|_{D_2}) = D_2$ by (e). From the efficiency of $c_1$ we know $0_1 \in c_1(\mathfrak{N}|_{D_1})$, and from $c_2(R^N_2) = D_2$ we know $a_2 \in c_2(\mathfrak{N}|_{D_2})$, so for any profile $P$ extending $\mathfrak{N}$, $(0_1, a_2) \in SbS(c_1, c_2)(P)$. By Lemma 11, $(d'_1, 0_2) \in Sbs(c_1, c_2)(P)$, which means $d'_1 \in c_1(Q^1)$. Therefore $c_1(Q^1) = D_1$.

Then we use $c_1(Q^1) = D_1$ to show $c_2(Q^2) = D_2$. If $c_2(Q^2) \neq D_2$, suppose $d_2 \in c_2(Q^2)$ and $e_2 \notin c_2(Q^2)$. First suppose $d < e$ (where $<$ is the usual strict order on integer numbers). We consider a collection of CP-nets $\mathfrak{N}' = (\mathcal{N}'_1, \ldots, \mathcal{N}'_N)$ s.t. for all $j \leq N$

$$\mathcal{N}'_j = (Q^1_j, Q^2_j).$$

By definition, $1_1 \succ_{Q^1_j} 2_1$ iff $Q^2_j = 0_2 \succ \ldots \succ a_2$, which holds iff $d_2 \succ_{Q^2_j} e_2$. Remember that $2_1 \in c_1(Q^1)$, therefore for any $P$ extending $\mathfrak{N}'$, $(2_1, d_2) \in SbS(c_1, c_2)(P)$. Notice that the preference between $(2_1, d_2)$ and $(1_1, e_2)$ is not determined because $2_1 \succ 1_1$ and $e_2 \succ d_2$, or $1_1 \succ 2_1$ and $d_2 \succ e_2$. By Lemma 11 we have $(1_1, e_2) \in SbS(c_1, c_2)(P)$. Therefore $e_2 \in c_2(Q^2)$, contradiction.

When $e_2 \rhd d_2$, consider the pair $(1_1, d_2, 0_3, \ldots, 0_p)$ and $(2_1, e_2, 0_3, \ldots, 0_p)$. Similarly there will be a contradiction. So $c_2(Q^2) = D_2$ in both cases. □

Now we conclude $c_2(Q^2) = D_2$, consider a set of CP-nets $\mathfrak{N}^* = (\mathcal{N}^*_1, \ldots, \mathcal{N}^*_N)$ s.t. for all $j \leq N$

$$\mathcal{N}^*_j = (V^1_j, Q^2_j, 0_3 \succ \ldots \succ a_3, \ldots, 0_k \succ \ldots \succ a_k).$$

By definition whenever $d_1 \succ_{V_j|_{D_1}} e_1$ we have $0_2 \succ_{V_j|_{D_2}} a_2$; whenever $d_1 \prec_{V_j|_{D_1}} e_1$ we have $a_2 \succ_{V_j|_{D_2}} 0_2$, so the preference between $(d_1, a_2, 0_3 \ldots, 0_p)$ and $(e_1, 0_2, 0_3 \ldots, 0_p)$ is not determined in any $\mathcal{N}_j$.

From Claim 1, $c_2(Q^2) = D_2$, therefore $a_2 \in c_2(Q^2)$. Furthermore, $d_1 \in c_1(V^1_1, \ldots, V^1_N)$. This allows us to say that for all $P$ extending $\mathfrak{N}^*$

$$(d_1, a_2, 0_3 \ldots, 0_p) \in SbS(c_1, \ldots, c_p)(P),$$

Therefore, from Lemma 11, we get that $(e_1, 0_2, 0_3 \ldots, 0_p) \in SbS(c_1, \ldots, c_p)(P)$, which implies that $e_1 \in c_1(P^1)$, contradicting the selection of $e_1$ at the beginning of the proof. This completes the proof. □

By Theorem 4, we know that in a neutral seat-by-seat voting rule/correspondence, either

1. each $r_i$ is efficient, or
2. each $r_i$ is anti-efficient.

The next lemma discusses a property about winners of a neutral seat-by-seat rule for the two cases separately.

**Lemma 16** *Suppose one of the following holds*

1. $p \geq 3$, *or*
2. $p = 2$ *and at least one of* $|D_1|$ *and* $|D_2|$ *is no less than 3.*

*Assume furthermore that* $SbS(c_1, \ldots, c_p)$ *is neutral. If there exists an N-vote profile* $P$ *s.t.* $SbS(c_1, \ldots, c_p)(P) \neq \mathcal{X}$, *then for all* $i \leq p$

1. $c_i(R_i^N) = \{0_i\}$, *when each* $r_i$ *is efficient*
2. $c_i(R_i^N) = \{a_i\}$, *when each* $r_i$ *is anti-efficient.*

**Proof of Lemma 16:** The proof in lemma 15 can be directly applied here. □

**Proof of Theorem 4:** In the proof of the theorem, we only give the proof for the case that **each** $r_i$ **is efficient**. The case that each $r_i$ is anti-efficient can be proved similarly.

If on the contrary there exists $i \leq p$ s.t. $c_i$ is not efficient, then there exists a profile $P^i = (V_1^i, \ldots, V_N^i)$ on $D_i$ and $d_1, e_1 \in D_i$ s.t. $d_1 \in c_i(P^i), e_1 \notin c_i(P^i)$, and $e_1 \succ_{V^i} d_1$ for all $V^i \in P^i$. W.l.o.g. we assume $i = 1$. We consider a set of CP-nets $\mathfrak{N} = (\mathcal{N}_1, \ldots, \mathcal{N}_N)$ s.t. for all $j \leq N$

1. $\mathcal{N}_j|_{D_1} = V_j^1$,
2. $\forall 2 \leq i \leq p, \mathcal{N}_j|_{D_i} = 0_i \succ \ldots \succ a_i$.

Consider the pair $(d_1, 0_2, 0_3 \ldots, 0_p)$ and $(e_1, a_2, 0_3, \ldots, 0_p)$. Notice for any $j \leq N$, $e_1 \succ_{V_j^1} d_1$ iff $0_2 \succ_{N_j|_{D_2}} a_2$, so by Lemma 10, the relative preference between $(d_1, 0_2, 0_3 \ldots, 0_p)$ and $(e_1, a_2, 0_3, \ldots, 0_p)$ is not determined. Remember we assume for all $2 \leq i \leq p$, $r_i$ is efficient, we always have $0_i \in c_i(\mathfrak{N}|_{D_i})$, so from Lemma 11 we know

$$(e_1, a_2, \ldots, 0_p) \in SbS(c_1, \ldots, c_p)(P).$$

which means $e_1 \in c_1(P^1)$, contradiction. The theorem is proved. □

**Proof of Theorem 5:** Again we prove the case that each $r_i$ is efficient, the anti-efficient case can be proved similarly.

The proof is similar with the proof of Theorem 1 and Theorem 2, which was presented in [4]. Assume that $SbS(c_1, \ldots, c_p)$ is not the trivial correspondence. If $SbS(c_1, \ldots, c_p)$ satisfies efficiency, then by Lemma 15 $c_i(R_i^0) = \{a_i\}$ for all $i \leq p$. If $SbS(c_1, \ldots, c_p)$ satisfies neutrality and local efficiency, then again, $c_i(R_i^0) = \{a_i\}$: Lemma 16 tells that $c_i(R_i^0) = \{a_i\}$ or $c_i(R_i^0) = \{0_i\}$, and the latter case is excluded by local efficiency. Define $k_i$ to be the constant that for all $l \leq k_i - 1$, $a_i \in c_i(R_i^l)$ and $a_i \notin c_i(R_i^{k_i})$. Since $a_i \notin c_i(R_i^N) = \{0_i\}$, $k_i$ is well defined and $k_i \leq N$ for all $i \leq p$.

First we claim $k_j = k_r$ for all $j, r \leq N$. If not, w.l.o.g. suppose $k_1 < k_2$. Consider CP-nets $\mathfrak{N} = (\mathcal{N}_1, \ldots, \mathcal{N}_p)$ s.t.

$$\begin{aligned} \mathcal{N}_1 = \mathcal{N}_2 = \ldots = \mathcal{N}_{k_1} : &\quad (0_1 \succ \ldots \succ a_1, 0_2 \succ \ldots \succ a_2, 0_3 \succ \ldots \succ a_3, \ldots, 0_p \succ \ldots \succ a_p) \\ \mathcal{N}_{k_1+1} = \ldots = \mathcal{N}_N : &\quad (a_1 \succ \ldots \succ 0_1, a_2 \succ \ldots \succ 0_2, 0_3 \succ \ldots \succ a_3, \ldots, 0_p \succ \ldots \succ a_p) \end{aligned}$$

By definition of $k_1$ and $k_2$, we know that $a_1 \notin c_1(R_1^{k_1})$ and $a_2 \in c_2(R_2^{k_1})$. Choose any $d_1 \in c_1(R_1^{k_1})$. By local efficiency, we have $(d_1, a_2, 0_3, \ldots, 0_p) \in SbS(c_1, \ldots, c_p)(P)$ for any $P$ extending $\mathfrak{N}$. Notice that in all $\mathcal{N}_j$, $d_1 \succ_{\mathcal{N}_j|_{D_1}} a_1$ iff $0_2 \succ_{\mathcal{N}_j|_{D_2}} a_2$, from Lemma 10, the preference between $(d_1, a_2, 0_3, \ldots, 0_p)$ and $(a_1, 0_2, 0_3, \ldots, 0_p)$ is not determined in any $\mathcal{N}_j$, so by Lemma 11

$$(a_1, 0_2, 0_3, \ldots, 0_p) \in SbS(c_1, \ldots, c_p)(P)$$

which means $a_1 \in c_1(R_1^{k_1})$, contradiction. Therefore $k_1 = k_2 = k$.

Now since $SbS(c_1, \ldots, c_p)$ is not a dictatorship, by Theorem 3 $c_i$ is not a dictatorship. So there exists an N-voter profile $P^i = (V_1^i, \ldots, V_N^i)$ on $D_i$ such that

$c_i(P^i) \neq \{top(V_k^i)\}$,because voter $k$ is not a dictator for $c_i$. Take $i=3$. We consider the following CP-nets $\mathfrak{N}' = (\mathcal{N}_1', \ldots, \mathcal{N}_N')$ where $\mathcal{N}_j'$ is defined by:

$$\mathcal{N}_j' : \begin{cases} (0_1 \succ \ldots \succ a_1, 0_2 \succ \ldots \succ a_2, V_j^3, \ldots, 0_p \succ \ldots \succ a_p) & \text{if } j \leq k-1 \\ (0_1 \succ \ldots \succ a_1, a_2 \succ \ldots \succ 0_2, V_j^3, \ldots, 0_p \succ \ldots \succ a_p) & \text{if } j = k \\ (a_1 \succ \ldots \succ 0_1, a_2 \succ \ldots \succ 0_2, V_j^3, \ldots, 0_p \succ \ldots \succ a_p) & \text{if } j \geq k+1 \end{cases}$$

Notice that $\mathcal{N}' = (R_1^k, R_2^{k-1}, P^3, R_4^N, \ldots, R_p^N)$. By definition of $k$, we have $a_1 \notin c_1(R_1^k)$, therefore there exists $d_1 \in c_1(\mathfrak{N}'|_{D_1})$ such that $d_1 \neq a_1$, and for any $P$ extending $\mathfrak{N}'$, any $d_3 \in c_3(P^3)$, we have, by local efficiency:

$$(d_1, a_2, d_3, 0_4, \ldots, 0_p) \in SbS(c_1, \ldots, c_p)(P).$$

Notice that when $j \neq k$, $d_1 \succ_{\mathcal{N}_j'|_{D_1}} a_1$ iff $0_2 \succ_{\mathcal{N}_j'|_{D_2}} a_2$. When $j = k$, $top(V_k^3) \succ_{\mathcal{N}_k'|_{D_3}} d_3$ for any $d_3 \in c_3(P^3)$. Therefore the relative preference between $(d_1, a_2, d_3, 0_4, \ldots, 0_p)$ and $(a_1, 0_2, top(V_k^3), 0_4, \ldots, 0_p)$ is not determined. Then by Lemma 11 we have

$$(a_1, 0_2, top(V_k^3), 0_4, \ldots, 0_p) \in SbS(c_1, \ldots, c_p)(P),$$

which means $a_1 \in c_1(R_1^k)$, contradicting with the selection of $k$. Theorem is thus proved. □

**Proof of Theorem 6:** Without loss of generality, let $|D_1| \geq 3$. This theorem is equivalent to *there is no non-trivial seat-by-seat and non-dictatorship voting correspondence of N-voters over $D_1 \times D_2$ that satisfies neutrality or efficiency.*

We prove this equivalent form instead. If it is not true, then there exists a non-trivial non-dictatorial $SbS(c_1, c_2)$ satisfying neutrality or efficiency. First we prove

**Claim 2** *Given $l \leq N$, let $(V_1^2, \ldots, V_N^2)$ be a profile on $D_2$ s.t.*

$$V_j^2 = \begin{cases} 0_2 \succ \ldots a_2 & \text{if } j = l \\ a_2 \succ \ldots 0_2 & \text{if } j \neq l \end{cases}.$$

*Then $c_2(V_1^2, \ldots, V_N^2) = \{a_2\}$.*

**Proof of Claim 2:** Without loss of generality, assume $l = 1$. In this case

$$V_j^2 = \begin{cases} 0_2 \succ \ldots \succ a_2 & \text{if } j = 1 \\ a_2 \succ \ldots \succ 0_2 & \text{if } j \geq 2 \end{cases}$$

and we write $P^2 = (V_1^2, \ldots, V_N^2)$. We first prove the following claim.

**Claim 3** *Let $P'^2 = (V_1'^2, \dots, V_N'^2)$ such that $V_1'^2 = V_1^2 = (0_2 \succ \dots \succ a_2)$, and for every $j \neq 2$, $V_j'^2 \in \{0_2 \succ \dots \succ a_2, a_2 \succ \dots \succ 0_2\}$. Denote the natural preorder on $D_2$ by $\rhd$, namely $0_2 \rhd \dots \rhd a_2$. Then*

*(1) if there exists $d_2 \neq a_2$ such that $d_2 \in c_2(P^2)$ then there exists $e_2 \unrhd d_2$ such that $e_2 \in c_2(P'^2)$.*

*(2) if $c_2(P'^2) = D_2$ then either $c_2(P^2) = \{a_2\}$ or $c_2(P^2) = D_2$.*

**Proof of Claim 3:** Without loss of generality we assume that $V_1'^2 = \dots = V_k'^2 = 0_2 \succ \dots \succ a_2$, and $V_{k+1}'^2 = \dots = V_N'^2 = a_2 \succ \dots \succ 0_2$. We prove (1) and (2) by induction on $k$. When $k = 1$, we have $P^2 = P'^2$ and (1) and (2) obviously hold. Suppose both propositions hold for $k = s$, and let us now show that they hold for $k = s+1$. Suppose there exists $d_2 \neq a_2$ such that $d_2 \in c_2(P^2)$, and consider the following $N$ CP-nets $\mathfrak{N} = (\mathcal{N}_1, \dots, \mathcal{N}_p)$

$$\begin{aligned} \mathcal{N}_1 = \dots = \mathcal{N}_s : &\qquad 0_1 \succ 1_1 \succ 2_1 \succ \dots \succ a_1, 0_2 \succ \dots \succ a_2. \\ \mathcal{N}_{s+1} : &\qquad 1_1 \succ 0_1 \succ 2_1 \succ \dots \succ a_1, a_2 \succ \dots \succ 0_2. \\ \mathcal{N}_{s+2} = \dots = \mathcal{N}_N : &\qquad 1_1 \succ 2_1 \succ 0_1 \succ \dots \succ a_1, a_2 \succ \dots \succ 0_2. \end{aligned}$$

Notice that

$$\mathcal{N}_j|_{D_2} = \begin{cases} 0_2 \succ \dots \succ a_2 & \text{if } j \leq s \\ a_2 \succ \dots \succ 0_2 & \text{if } j > s \end{cases}$$

So for any $Q = (W_1, \dots, W_N)$ such that $W_j$ extends $\mathcal{N}_j$, by induction hypothesis there exists $e_2 \in c_2(Q|_{D_2})$ and $e_2 \neq a_2$. Notice that in each $\mathcal{N}_j$, $0_1 \succ 1_1$ iff $e_2 \succ a_2$, so the preference between $(0_1, a_2)$ and $(1_1, e_2)$ are not determined. If $1_1 \in c_1(Q|_{D_1})$ then $(1_1, e_2) \in SbS(c_1, c_2)(Q)$, because $e_2 \in c_2(Q|_{D_2})$, and then by lemma 11, $(0_1, a_2) \in SbS(c_1, c_2)(Q)$, therefore $0_1 \in c_1(Q|_{D_1})$. On the other hand, if $2_1 \in c_1(Q|_{D_1})$, by local efficiency (either explicitly assumed or obtained as a consequence of global efficiency) we know that $1_1 \in c_1(Q|_{D_1})$, which implies that $0_1 \in c_1(Q|_{D_1})$. If for any $f \geq 3$, $f_1 \in c_1(Q|_{D_1})$, then again by local efficiency we get $0_1 \in c_1(Q|_{D_1})$. Therefore, we conclude that $0_1 \in c_1(Q|_{D_1})$.

Now consider another collection of $N$ CP-nets $\mathcal{N}_1' \dots, \mathcal{N}_N'$ s.t.

$$\begin{aligned} \mathcal{N}_1' = \dots = \mathcal{N}_s' : &\qquad 0_1 \succ 1_1 \succ 2_1 \succ \dots \succ a_1, 0_2 \succ \dots \succ a_2. \\ \mathcal{N}_{s+1}' : &\qquad 1_1 \succ 0_1 \succ 2_1 \succ \dots \succ a_1, 0_2 \succ \dots \succ a_2. \\ \mathcal{N}_{s+2}' = \dots = \mathcal{N}_N' : &\qquad 1_1 \succ 2_1 \succ 0_1 \succ \dots \succ a_1, a_2 \succ \dots \succ 0_2. \end{aligned}$$

We have $\mathcal{N}'_j = \mathcal{N}_j$ for all $j \neq s+1$, and $\mathcal{N}'_{s+1}|_{D_1} = \mathcal{N}_{s+1}|_{D_1}$. So for any profile $Q'$ extending these CP-nets, we have $0_1 \in c_1(Q|_{D_1}) = c_1(Q'|_{D_1})$. Now, two cases:

- either $c_2(Q'|_{D_2})$ contains some $f_2$ such that $d_2 \rhd f_2$. Notice then that in any $\mathcal{N}'_j$, $0_1 \succ 2_1$ holds if and only if $d_2 \succ f_2$ holds, so by Lemma 10 the preference between $(0_1, f_2)$ and $(2_1, d_2)$ is not determined, and by Lemma 11 we get $d_2 \in c_2(Q'|_{D_2})$, and (1) holds.

- or $c_2(Q'|_{D_2})$ contains some $f_2$ such that $f_2 \unrhd d_2$ and in this case (1) holds as well.

Since $Q'|_{D_2}$ corresponds to the case $k = s+1$, we conclude that (1) holds for $k = s+1$.

To see that (2) holds for $k = s+1$, suppose $d_2 \in c_2(P^2)$ and $c_2(Q'|_{D_2}) = D_2$, we will show $c_2(P^2) = D_2$. We still consider the two collections of CP-nets $\mathcal{N}_j$ $(j \leq N)$ and $\mathcal{N}'_j$ $(j \leq N)$. Since in any $\mathcal{N}'_j$, we have $0_1 \succ 2_1$ if and only if $0_2 \succ a_2$, by Lemma 10 the preference between $(0_1, a_2)$ and $(2_1, 0_2)$ is not determined in any $\mathcal{N}'_j$, and we just proved in the induction step to prove (1) that if $d_2 \in c_2(P^2)$ then $0_1 \in Q'|_{D_1}$. So $(0_1, a_2) \in SbS(c_1, c_2)(Q')$. From Lemma 11 we know $(2_1, 0_2) \in SbS(c_1, c_2)(Q')$, which means $2_1 \in Q'|_{D_1}$, and then $1_1 \in Q'|_{D_1}$ by efficiency of $c_1$. Therefore $\{0_1, 1_1, 2_1\} \in Q|_{D_1}$.

Then choose any $e_2 \in c_2(Q|_{D_2})$. Because $\{0_1, 1_1, 2_1\} \in Q|_{D_1}$,

$$(0_1, e_2) \in SbS(c_1, c_2)(Q) \text{ and } (1_1, e_2) \in SbS(c_1, c_2)(Q).$$

For any $f_2 \rhd e_2$, the preference between $(0_1, e_2)$ and $(1_1, f_2)$ is not determined in each $\mathcal{N}_j$, so by Lemma 11 $f_2 \in c_2(Q|_{D_2})$. Similarly when $e_2 \rhd f_2$ we consider the pair $(0_1, f_2)$ and $(1_1, e_2)$ and conclude $f_2 \in c_2(Q|_{D_2})$. Therefore $c_2(Q|_{D_2}) = D_2$. By induction hypothesis we know $c_2(P^2) = D_2$. So (2) holds for $k = s+1$.

From the above steps we know (1) and (2) hold for all $k$. Claim 3 is proved. □

Then we come back to Claim 2 and use Claim 3 to prove it. Since $c_1$ is not a dictatorship, there exists a profile $Q^1 = (W_1^1, \ldots, W_N^1)$ on $D_1$ s.t. there exists $d_1 \in c_1(Q^1)$ s.t. $d_1 \neq e_1 = (W_1^1)_1$. Construct CP-nets $\mathfrak{N} = (\mathcal{N}_1, \ldots, \mathcal{N}_p)$ s.t. for all $j \leq N$

1. $\mathcal{N}_j|_{D_1} = W_j^1$,

2. $\mathcal{N}_j|_{D_2} = \begin{cases} 0_2 \succ \ldots \succ a_2 & \text{if } e_1 \succ_{W_j^1} d_1 \\ a_2 \succ \ldots \succ 0_2 & \text{if } d_1 \succ_{W_j^1} e_1 \end{cases}$

We are going to prove that $c_2(P^2) = \{a_2\}$. If $c_2(P^2) \neq \{a_2\}$, choose any $d_2 \in c_2(P^2)$ s.t. $d_2 \neq a_2$. Notice that $\mathcal{N}_1|_{D_2} = 0_2 \succ \ldots \succ a_2$, thus by Claim 3 there exists $e_2 \trianglerighteq d_2$ and $e_2 \in c_2(\mathfrak{N}|_{D_2})$. So for all profile $P$ extending $\mathfrak{N}$, $(d_1, e_2) \in SbS(c_1, c_2)(P)$. First look at the pair $(d_1, e_2)$ and $(e_1, a_2)$. Notice in every $\mathcal{N}_j$, $d_1 \succ e_1$ iff $a_2 \succ e_2$, so by Lemma 10 their relative preference is not determined in any $\mathcal{N}_j$, so by Lemma 11 $e_1 \in c_1(\mathfrak{N}|_{D_1}), a_2 \in c_2(\mathfrak{N}|_{D_2})$. Then for any $h_2 \in D_2$, $h_2 \neq a_2$, consider the pair $(e_1, a_2)$ and $(d_1, h_2)$, whose relative preference is not determined either. From Lemma 11 we get that $h_2 \in c_2(\mathfrak{N}|_{D_2})$, which implies $c_2(\mathfrak{N}|_{D_2}) = D_2$. Then by Claim 3(2), $c_2(P^2) = D_2$. We consider the following CP-nets $\mathfrak{N}'$

$$\begin{array}{rl} \mathcal{N}_1' : & 0_1 \succ 2_1 \succ 1_1 \succ \ldots \succ a_1, 0_2 \succ \ldots \succ a_2 \\ \mathcal{N}_2' \sim \mathcal{N}_N' : & 1_1 \succ 0_1 \succ 2_1 \succ \ldots \succ a_1, a_2 \succ \ldots \succ 0_2 \end{array}$$

Notice that in any $\mathcal{N}_j'$, $0_1 \succ 1_1$ iff $0_2 \succ a_2$, $1_1 \succ 2_1$ iff $a_2 \succ 0_2$, by Lemma 10, in every $\mathcal{N}_j'$ the relative preference of the following two pairs is not determined : $(0_1, a_2)$ and $(1_1, 0_2)$, $(2_1, a_2)$ and $(1_1, 0_2)$. Notice $\mathfrak{N}'|_{D_2} = P^2$, so $SbS(c_1, c_2)(\mathfrak{N}'|_{D_2}) = D_2$. Applying Lemma 11, we get

(a) if $0_1 \in c_1(\mathfrak{N}'|_{D_1})$, then $1_1 \in c_1(\mathfrak{N}'|_{D_1})$;

(b) if $1_1 \in c_1(\mathfrak{N}'|_{D_1})$, then $2_1 \in c_1(\mathfrak{N}'|_{D_1})$.

And from the efficiency of $c_1$ we get

(c) if $2_1 \in c_1(\mathfrak{N}'|_{D_1})$, then $0_1 \in c_1(\mathfrak{N}'|_{D_1})$ (because of the local efficiency and $0_1 \succ_i 2_1$ for every $i$)

(d) if for $d_1 \notin \{0_1, 1_1, 2_1\}$, $d_1 \in c_1(\mathfrak{N}'|_{D_1})$, then $\{0_1, 1_1, 2_1\} \subseteq c_1(\mathfrak{N}'|_{D_1})$.

From (a), (b) and (c) we get that either $c_1(\mathfrak{N}'|_{D_1})$ contains $0_1$, $1_1$ and $2_1$, or does not contain any of them. Because $c_1(\mathfrak{N}'|_{D_1}) \neq \emptyset$, if it does not contain any of them then it contains some $d_1 \notin \{0_1, 1_1, 2_1\}$ and by (d) we get that $\{0_1, 1_1, 2_1\} \subseteq c_1(\mathfrak{N}'|_{D_1})$ as well. Therefore, in all cases, $\{0_1, 1_1, 2_1\} \subseteq c_1(\mathfrak{N}'|_{D_1})$. We next consider the following CP-nets $\mathfrak{N}^*$

$$\mathcal{N}_1^*: \quad 0_1 \succ 2_1 \succ 1_1 \succ \ldots \succ a_1, 0_2 \succ \ldots \succ a_2$$
$$\mathcal{N}_2^* \sim \mathcal{N}_N^*: \quad 1_1 \succ 0_1 \succ 2_1 \succ \ldots \succ a_1, 0_2 \succ \ldots \succ a_2$$

Namely $\mathfrak{N}^*|_{D_2} = R_2^N$, and $\mathfrak{N}^*|_{D_1} = \mathfrak{N}'|_{D_1}$. So $\{0_1, 1_1, 2_1\} \subseteq c_1(\mathfrak{N}^*|_{D_1})$. Choose any $d_2 \in c_2(R_2^N)$ and for any $P$ extending $\mathfrak{N}^*$, $(0_1, d_2) \in SbS(c_1, c_2)(P)$ and $(2_1, d_2) \in SbS(c_1, c_2)(P)$. Then for any $e_2$ s.t. $e_2 \rhd d_2$ we consider the pair $(0_1, d_2)$ and $(2_1, e_2)$; for any $d_2 \rhd e_2$ we consider the pair $(0_1, e_2)$ and $(2_1, d_2)$. Similarly by Lemma 10 and Lemma 11, in either case $e_2 \in c_2(R_2^N)$. So $c_2(R_2^N) = D_2$. Remember $SbS(c_1, c_2)$ is non-trivial, therefore by Lemma 15 $c_2(R_2^N) = \{0_2\}$, which is a contradiction. So $c_2(P^2) = \{a_2\}$, Claim 2 is proved. □

When there are two voters, notice that in the proof of Claim 2 the names of the candidates are not relevant, so if we exchange $(0_2 \succ \ldots \succ a_2)$ and $(a_2 \succ \ldots \succ 0_2)$, a similar claim holds. But then $c_2(0_2 \succ \ldots \succ a_2, a_2 \succ \ldots \succ 0_2) = \{a_2\} = \{0_2\}$, a contradiction.

Then we consider $N \geq 3$. Based on Claim 3, we will derive the following claim, which will then lead to a contradiction.

**Claim 4** *For any* $1 \leq k \leq N$, $c_2(R_2^k) = \{a_2\}$.

**Proof of Claim 4:** We prove this by induction on $k$. When $k = 1$, it is implied by Claim 2. Suppose that the claim holds for $k = s$, and let now $k = s+1$. Define $P^1 = (V_1^1, \ldots, V_N^1)$ s.t.

$$V_j^1 = \begin{cases} 0_1 \succ 1_1 \succ 2_1 \succ \ldots \succ a_1 & \text{if } j \leq s \\ 2_1 \succ 0_1 \succ 1_1 \succ \ldots \succ a_1 & \text{if } j = s+1 \\ 1_1 \succ 2_1 \succ 0_1 \succ \ldots \succ a_1 & \text{if } j \geq s+2 \end{cases}$$

We claim $c_1(P^1) = \{1_1\}$. To see $2_1 \notin c_1(P^1)$, consider the following CP-nets $\mathfrak{N}$ s.t. $\mathfrak{N}|_{D_1} = P^1$.

$$\mathcal{N}_1 = \ldots = \mathcal{N}_s: \quad 0_1 \succ 1_1 \succ 2_1 \succ \ldots \succ a_1, a_2 \succ \ldots \succ 0_2$$
$$\mathcal{N}_{s+1}: \quad 2_1 \succ 0_1 \succ 1_1 \succ \ldots \succ a_1, 0_2 \succ \ldots \succ a_2$$
$$\mathcal{N}_{s+2} = \ldots = \mathcal{N}_N: \quad 1_1 \succ 2_1 \succ 0_1 \succ \ldots \succ a_1, a_2 \succ \ldots \succ 0_2$$

By Claim 2, $c_2(\mathfrak{N}|_{D_2}) = \{a_2\}$.

Assume $2_1 \in c_1(P^1)$, therefore $(2_1, a_2) \in SbS(c_1, c_2)(\mathcal{N})$. For any $j$ we have $2_1 \succ_{\mathcal{N}_j} 1_1$ if and only if $0_2 \succ_{\mathcal{N}_j} a_2$, therefore the preference between $(2_1, a_2)$ and $(1_1, 0_2)$ is not determined in any of the $\mathcal{N}_j$'s, so $0_2 \in c_2(\mathcal{N}_p^2|_{D_2})$, which contradicts Claim 2. Clearly there is no $d_1 \notin \{0_1, 1_1, 2_1\}$ and $d_1 \in c_1(P^1)$, otherwise by efficiency $2_1 \in c_1(P^1)$.

To see $0_1 \notin c_1(P^1)$, consider the following CP-nets $\mathfrak{N}'$ s.t. $\mathfrak{N}'|_{D_1} = P^1$, $\mathfrak{N}'|_{D_2} = R_2^s$.

$$\begin{aligned} \mathcal{N}_1' = \ldots = \mathcal{N}_s' : &\quad 0_1 \succ 1_1 \succ 2_1 \succ \ldots \succ a_1, 0_2 \succ \ldots \succ a_2 \\ \mathcal{N}_{s+1}' : &\quad 2_1 \succ 0_1 \succ 1_1 \succ \ldots \succ a_1, a_2 \succ \ldots \succ 0_2 \\ \mathcal{N}_{s+2}' = \ldots = \mathcal{N}_N' : &\quad 1_1 \succ 2_1 \succ 0_1 \succ \ldots \succ a_1, a_2 \succ \ldots \succ 0_2 \end{aligned}$$

By the induction hypothesis, $c_2(\mathfrak{N}'|_{D_2}) = \{a_2\}$. If $0_1 \in c_1(P^1)$, then $SbS(c_1, c_2)(\mathcal{N}')$ contains $(0_1, a_2)$. Consider the pair $(0_1, a_2)$ and $(2_1, 0_2)$, by Lemma 11, $0_2 \in c_2(\mathfrak{N}'|_{D_2})$, contradiction. Therefore $c_1(P^1) = \{1_1\}$.

Lastly we consider the following CP-nets $\mathfrak{N}^*$ s.t. $\mathfrak{N}^*|_{D_1} = P^1$, $\mathfrak{N}^*|_{D_2} = R_2^{s+1}$

$$\begin{aligned} \mathcal{N}_1^* = \ldots = \mathcal{N}_s^* : &\quad 0_1 \succ 1_1 \succ 2_1 \succ \ldots \succ a_1, 0_2 \succ \ldots \succ a_2 \\ \mathcal{N}_{s+1}^* : &\quad 2_1 \succ 0_1 \succ 1_1 \succ \ldots \succ a_1, 0_2 \succ \ldots \succ a_2 \\ \mathcal{N}_{s+2}^* = \ldots = \mathcal{N}_N^* : &\quad 1_1 \succ 2_1 \succ 0_1 \succ \ldots \succ a_1, a_2 \succ \ldots \succ 0_2 \end{aligned}$$

If there exists $d_2 \in c_2(\mathfrak{N}^*|_{D_2})$ such that $d_2 \neq a_2$, then $(1_1, d_2) \in SbS(c_1, c_2)(\mathcal{N}^*)$. Then consider $(1_1, d_2)$ and $(0_1, a_2)$. Notice that in $\mathcal{N}_j^*$, $1_1 \succ 0_1$ if and only if $a_2 \succ d_2$, therefore the preference between $(1_1, d_2)$ and $(0_1, a_2)$ is not determined in any of the $\mathcal{N}_j^*$, so by Lemma 10 their relative preference is not determined in any $\mathcal{N}_j^*$. So from $(1_1, d_2) \in SbS(c_1, c_2)(P^*)$ where $P^*$ extends $\mathfrak{N}^*$, by Lemma 11, $0_1 \in c_1(P^1)$, contradiction. Therefore, $c_2(R_2^{s+1}) = c_2(\mathfrak{N}^*|_{D_2}) = \{a_2\}$, and the claim holds when $k = s+1$, which completes the proof. □

By Claim 4, when $k = N$, we have $c_2(R_2^N) = \{a_2\}$. Because the premise of Lemma 15 is satisfied by any profile $P$ whose projection on $D_2$ is $R_2^N$, from Lemma 15 we get $c_2(R_2^N) = \{0_2\}$, a contradiction. So the assumption in the beginning of the proof is not true, which means the theorem is true. □