

New Candidates Welcome!

Possible Winners with respect to the Addition of New Candidates

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Abstract

In some voting contexts, some new candidates may show up in the course of the process. In this case, we may want to determine which of the initial candidates are possible winners, given that a fixed number k of new candidates will be added. We give a computational study of the latter problem, focusing on scoring rules, and we give a formal comparison with related problems such as control via adding candidates or cloning.

Key words: Computational social choice

1. Introduction

In many real-life collective decision making situations, the set of candidates (or alternatives) may vary while the voting process goes on, and may change at any time before the decision is final: some new candidates may join, whereas some others may withdraw. This, of course, does not apply to situations where the vote takes place in a very narrow period of time (such as, typically, political elections in most countries), and the addition of new candidates during the process does not apply either to situations where the law forbids new candidates to be introduced after the vote has started (which, again, is the case for most political elections). However, there are quite many practical situations where this situation does happen, especially contexts where votes are sent by email during an extended period of time. This is typically the case when making a decision about the date and time of a meeting. In the course of the process, we may learn that the room

is taken at a given time slot, making this time slot no longer a candidate. The opposite case also occurs frequently; we thought the room was taken on a given date and then we learn that it has become available, making this time slot a new candidate.

The paper focuses on candidate addition only. More precisely, the class of situations we consider is the following. A set of voters have expressed their votes about a set of (initial) candidates. Then some new candidates declare. The winner will ultimately be determined using some given voting rule. In this class of situations, an important question arises: *who among the initial candidates can still be a winner once the voters' preferences about all candidates are known?* This is important in particular if there is some interest to detect as soon as possible the candidates who are not possible winners: for instance, candidates for a job may have the opportunity to apply for different positions, and time slots may be released for other potential meetings.

This question is strongly related to several streams of work from the recent literature on computational social choice, especially the problem of determining whether the vote elicitation process is terminated [8, 21]; the possible winner problem, and more generally the problem of applying a voting rule to incomplete preferences [15, 18, 22, 4, 5] or uncertain preferences with probabilistic information [13]; swap bribery, encompassing the possible winner problem as a particular case [10]; and finally, to the control of a voting rule by the chair via adding candidates and to resistance to cloning—we shall come back to these two latter problems in more detail in the related work section.

Clearly, considering situations where new voters are added is a specific case of voting under incomplete preferences, where incompleteness is of a very specific type: the set of candidates is partitioned in two groups (the initial and the new candidates), and the incomplete preferences consist of complete rankings on the initial candidates. This class of situations is somehow dual of a class of situations that has been considered more often, namely, when the set of *voters* is partitioned in two groups: those voters who have already voted, and those who haven't expressed their votes yet. The latter class of situations, while being a subclass of voting under incomplete preferences, has been more specifically studied as a *coalitional manipulation problem* [7, 24], where the problem is to determine whether it is possible for the voters who haven't voted yet to make a given candidate win. Varying sets of voters have also been studied in the context of compiling the votes of a subelectorate [6, 23]: there, one is interested in synthesizing a set of initial votes, while still being able to compute the outcome once the remaining voters have expressed their votes.

The layout of the paper is as follows. In Section 2 we recall the necessary background on voting and we introduce a few notations. In Section 3 we state the problem formally, by defining voting situations where candidates may be added after the votes over a subset of initial candidates have been already elicited. In the following sections we focus on specific voting rules and we study the problem from a computational point of view. In Section 4, we focus on the family of K -approval rules, including plurality and veto as specific subcases, and give a full dichotomy result as to the complexity of the possible winner with respect to the addition of k new candidates; namely, we show that the problem is NP-complete as soon as $K \geq 3$ and $k \geq 3$, and polynomial if $K \leq 2$ or $k \leq 2$. In Section 5 we focus on the Borda rule and show that the problem is polynomial regardless of the number of new candidates, and we exhibit a more general family of voting rules, including the Borda rule, for which this result can be generalized. In Section 6 we show that the problem can be hard for some positional scoring rules even if only one new candidate is added. In Section 7 we discuss the relationship to the general possible winner problem, to the control of an election by the chair via adding candidates, and to candidate cloning. Section 8 summarizes the results and mention further research directions.

2. Background and notations

Let C be a finite set of *candidates*, and N a finite set of *voters*. Let $p = |C|$ and $n = |N|$. A C -*vote* (called more simply a vote when this is not ambiguous) is a linear order over C . We sometimes denote votes in the following way: $a \succ b \succ c$ is denoted by abc , etc. A n -voter C -*profile* is a collection $P = \langle V_1, \dots, V_n \rangle$ of C -votes. Let \mathcal{P}_C be the set of all C -votes and therefore \mathcal{P}_C^n be the set of all n -voter C -profiles.

A voting rule on C is a function r from \mathcal{P}_C^n to C . As the usual definition of most voting rules does not exclude the possibility of ties, it is usually assumed that these ties are broken by a fixed priority order on candidates. In this paper, we don't fix a priority order on candidates (one reason being that the complete set of candidates is not known to start with), which means that we consider voting correspondences (*i.e.*, functions from \mathcal{P}_C^n to $2^C \setminus \{\emptyset\}$) rather than voting rules and ask whether x is a *possible cowinner* for a given profile P . This is equivalent to asking whether there exists a priority order for which x is a possible winner, or else whether x is a possible winner for the most favorable priority order (with x having priority over all other candidates). With a slight abuse of notation we

denote voting correspondence by r just as voting rules, and $r(P)$ is the set of *co-winners* for profile P .

A voting rule (resp. correspondence) is anonymous if it insensitive to the identity of voters, and neutral if it insensitive to the identity of candidates.

For $P \in \mathcal{P}_C^n$ and $x, x' \in C$, let $n(P, i, x)$ be the number of votes in P ranking x in position i , $ntop(P, x) = n(P, 1, x)$ the number of votes in P ranking x first, and $N_P(x, x')$ the number of votes in P ranking x above x' . Let $\vec{s} = \langle s_1, \dots, s_p \rangle$ be a vector of integers such that $s_1 \geq \dots \geq s_p$ and $s_1 > s_p$. The scoring rule $r_{\vec{s}}(P)$ induced by \vec{s} elects the candidate maximizing $score_{\vec{s}}(x, P) = \sum_{i=1}^p s_i \cdot n(P, i, x)$. Again, in case of a tie, a priority relation on candidates is applied.

If K is a fixed integer then K -approval, r_K , is the scoring rule corresponding to the vector $s_K = \langle 1, \dots, 1, 0, \dots, 0 \rangle$ – with K 1's and $p - K$ 0's. The K -approval score $score_{s_K}(x, P)$ of a candidate x is denoted more simply by $S_K^P(x)$: in other words, $S_K^P(x)$ is the number of voters in P who rank x in the K first positions, *i.e.*, $S_K^P(x) = \sum_{i=1, \dots, K} n(P, i, x)$. When $K = 1$, we get the *plurality* rule r_P , and when $K = p - 1$ we get the *veto* (or *antiplurality*) rule. The *Borda* rule r_B is the scoring rule corresponding to the vector $\langle p - 1, p - 2, \dots, 0 \rangle$.

We now define formally situations where new candidates are added.

Definition 1 A voting situation with a varying set of candidates is a 4-uple $\Sigma = \langle N, X, P_X, k \rangle$ where N is a set of voters (with $|N| = n$), X a set of candidates, $P_X = \langle V_1, \dots, V_n \rangle$ a n -voter X -profile, and k is a positive integer.

X denotes the set of initial candidates, P_X the initial profile, and k the number of new candidates. Nothing is known a priori about the voters' preferences relatively to the new candidates, henceforth their identity is irrelevant and only their number counts.

Because the number of candidates is not the same before and after the new candidates come in, we have to consider families of voting rules (for a varying number of candidates) rather than voting rules for a fixed number of candidates. While it is true that for many usual voting rules there is an obvious way of having them defined for a varying number of candidates, this is not the case for all of them, especially scoring rules other than plurality, Borda and veto. We shall therefore consider *collections of voting rules*, parameterized by the number of candidates. We slightly abuse notation and denote these collections of voting rules by r . Again with a slight abuse of notation, we often write $r(P)$ instead of $r_p(P)$. The complexity results we give in this paper bear on such collections of voting rules, where the number of candidates is variable.

If P is a C -profile and $C' \subseteq C$, then the projection of P on C' , denoted by $P^{\downarrow C'}$, is obtained by deleting all candidates in $C \setminus C'$ in each of the votes of P , and leaving unchanged the ranking on the candidates of C' . For instance, let us take $P = \langle abcd, dcab \rangle$ then $P^{\downarrow\{a,b\}} = \langle ab, ab \rangle$ and $P^{\downarrow\{a,b,c\}} = \langle abc, cab \rangle$. In all situations, the set of initial candidates is denoted by $X = \{x_1, \dots, x_p\}$, the set of the k new candidates is denoted by $Y = \{y_1, \dots, y_k\}$. If P_X is a X -profile and P a $X \cup Y$ -profile, then we say that P extends P_X if the projection of P on the candidates in X is exactly P_X . For instance, let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2\}$; the profile $P = \langle x_1y_1x_2y_2x_3, y_1y_2x_1x_2x_3, x_3x_2y_2y_1x_1 \rangle$ extends the X -profile $P = \langle x_1x_2x_3, x_1x_2x_3, x_3x_2x_1 \rangle$.

3. Possible winners when new candidates are added

We recall from [15] that given a collection $\langle P_1, \dots, P_n \rangle$ of partial strict orders on C representing some incomplete information about the votes, a candidate x is a possible winner if there is a profile $\langle T_1, \dots, T_n \rangle$ where each T_i is a ranking on C extending P_i in which x wins. Reformulated in the case where P_i is a ranking of the initial candidates (those in X), we get the following definition:

Definition 2 *Given a voting situation $\Sigma = \langle N, X, P_X, k \rangle$, a set of new candidates Y (with $|Y| = k$), and a collection r of voting rules, we say that $x^* \in X$ is a possible co-winner with respect to Σ , Y , and r if there is a $X \cup Y$ -profile P extending P_X such that $x^* \in r(P)$.*

We can also define the notion of *necessary co-winner* with respect to Σ and r : $x^* \in X$ is a *necessary co-winner with respect to Σ , Y , and r* if for every $X \cup Y$ -profile P extending P_X we have $x^* \in r(P)$. However, the study of necessary winners in this particular setting will almost never lead to any significant result. Indeed, as soon as we have neutrality, all new candidates are necessary co-winners. There may be necessary co-winners among the initial candidates, but again this will happen rarely (and the case will be discussed for a few specific voting rules in the corresponding parts of the paper).

Now we are in position to consider specific voting rules.

4. K -approval

As a warm-up we start by plurality and veto, even though the results we give are subsumed by further results, because it provides some important intuitions.

4.1. Plurality and veto

Let us start with an example: suppose $X = \{a, b, c\}$, $n = 13$, and the plurality scores in P_X are $a \mapsto 6$, $b \mapsto 4$, $c \mapsto 2$. There is only one new candidate (y). We have that:

1. a is a possible winner (a will win in particular if the top candidate of every voter remains the same);
2. b is a possible winner: to see this, suppose that 2 voters who had ranked a first now rank y first; the new scores are $a \mapsto 4$, $b \mapsto 4$, $c \mapsto 3$, $y \mapsto 2$;
3. c is not a possible winner: to make her having at least the same score than that of a and b , we need at least 3 (resp. 1) voters who had ranked a (resp. b) first now rank y first; but this then means that y gets at least 4 votes, while c has only 3.

More generally, we have the following result:

Proposition 1 *Let P_X be an n -voter profile on X , and $x \in X$. The candidate x^* is a possible cowinner for P_X and plurality with respect to the addition of k new candidates if and only if*

$$ntop(P_X, x^*) \geq \frac{1}{k} \cdot \sum_{x_i \in X} \max(0, ntop(P_X, x_i) - ntop(P_X, x^*))$$

Proof: Suppose first that the inequality holds. We build the following $(X \cup Y)$ -profile P extending P_X :

1. for every candidate x_i such that $ntop(P_X, x_i) > ntop(P_X, x^*)$ we simply take $ntop(P_X, x_i) - ntop(P_X, x^*)$ arbitrary votes ranking x_i on top and place one of the y_j 's on top of the vote (and the other y_j 's anywhere), subject to the condition that no y_j is placed on top of a vote more than $ntop(P_X, x^*)$ times. (This is possible because the inequality is satisfied).
2. in all other votes (those not considered at step 1), place all y_j 's anywhere except on top.

We obtain a profile P extending P_X . First, we have $ntop(P, x^*) = ntop(P_X, x^*)$, because on all the votes in P_X where x is on top, the new top candidate in the corresponding vote in P is still x^* (cf. step 2), and all the votes in P_X where x^* was not on top obviously cannot have x^* on top in the corresponding vote in P . Second, let $x_i \neq x^*$. If $ntop(P_X, x_i) \leq ntop(P_X, x^*)$ then $ntop(P, x_i) = ntop(P_X, x_i)$; and if $ntop(P_X, x_i) > ntop(P_X, x^*)$ then we have

$ntop(P, x_i) = ntop(P_X, x_i) - (ntop(P_X, x_i) - ntop(P_X, x^*)) = ntop(P_X, x^*)$.

Therefore, the winner for plurality in P is x^* .

Conversely, if the inequality is not satisfied, in order for x^* to become the winner in P , the other x_i 's must lose globally an amount of $\sum_{x_i \in X} \max(0, ntop(P_X, x_i) - ntop(P_X, x^*))$ votes; and since $\sum_{x_i \in X} \max(0, ntop(P_X, x_i) - ntop(P_X, x^*)) > k \cdot ntop(P_X, x^*)$, for at least one of the y_j 's we will have $ntop(P, y_j) > ntop(P, x^*)$; therefore x^* cannot be the winner for plurality in P . ■

For veto, this is almost trivial. By placing any of the new candidates below x^* in every vote of P_X where x^* is ranked at the bottom position, we obtain a vote P where no one vetoes x^* , so any candidate is a possible co-winner.

Therefore, computing possible (co-)winners for plurality and veto with respect to candidate addition is polynomial (which we already knew, since possible winners for plurality and veto can be computed in polynomial time [4]).

4.2. K -approval, one new candidate

Recall that we denote by $S_K(P_X, x_j)$ the score of x_j for P_X and K -approval (*i.e.* the number of voters who rank x_j among their top K candidates); and by $n(P_X, K, x_j)$ the number of voters who rank x_j exactly in position K . We start by the case where a single candidate is added.

Proposition 2 *Let $1 \leq K \leq p - 1$, P_X an n -voter profile on X , and $x \in X$. A candidate x^* is a possible cowinner for P_X and K -approval with respect to the addition of one new candidate if and only if the following two conditions hold:*

1. *for any $x_i \neq x$, if $S_K(P_X, x_i) > S_K(P_X, x^*)$
then $n(P_X, K, x_i) \geq S_K(P_X, x_i) - S_K(P_X, x^*)$.*
2. $S_K(P_X, x^*) \geq \sum_{x_i \in X} \max(0, S_K(P_X, x_i) - S_K(P_X, x^*))$

Proof: Assume conditions (1) and (2) are satisfied. then we build the following $(X \cup Y)$ -profile extending P_X :

- for every x_i such that $S_K(P_X, x_i) > S_K(P_X, x^*)$, we take $S_K(P_X, x_i) - S_K(P_X, x^*)$ arbitrary votes who rank x_i in position K in P_X and place y on top. This is possible because condition (1) is satisfied.
- in all other votes (those not considered at step 1), place all y at the bottom.

We obtain a profile P extending P_X . First, we have $S_K(P, x^*) = S_K(P_X, x^*)$, because (a) all votes in P_X ranking x^* in position K are extended in such a way that all y is placed on the bottom position, therefore x^* gets a point in each of these votes if and only if it got a point in P_X , and (b) all votes ranking x^* in position other than K in P_X get a point in P if and only if they get a point in P , both in the case y was added on top of the vote and in the case it was added on bottom of the vote. Second, for every x_i such that $S_K(P_X, x_i) > S_K(P_X, x^*)$, x_i loses exactly $S_k(P_X, x_i) - S_k(P_X, x^*)$ points when P_X is extended into P , therefore $S_K(P, x_i) = S_K(P_X, x_i) - S_K(P_X, x_i) + S_K(P_X, x^*) = S_K(P_X, x^*)$. Third, $S_K(P, y) = \sum_{x_i \in X \setminus \{x_i\}} \max(0, S_K(P_X, x_i) - S_K(P_X, x^*)) \leq S_K(P_X, x^*)$ – because of (2) – hence $S_K(P, y) \leq S_K(P, x^*)$. Therefore, x^* is a cowinner for K -approval in P .

Now, assume condition (1) is not satisfied, that is, there is a x_i such that $S_K(P_X, x_i) > S_K(P_X, x^*)$ and such that $n(P_X, K, x_i) < S_K(P_X, x_i) - S_K(P_X, x^*)$. There is no way of having x_i losing more than $Z_K(P_X, x_i)$ points, therefore x^* will never catch up x_i 's advance and is therefore not a possible winner. Finally, assume condition (2) is not satisfied, which means that we have $\sum_{x_i \in X \setminus \{x_i\}} \max(0, S_K(P_X, x_i) - S_K(P_X, x^*)) > S_K(P_X, x^*)$. Then, to catch up the advance of the x_i 's on x^* we must add y in one of the top K position in a number of votes exceeding $S_K(P_X, x^*)$, therefore $S_K(P, y) > S_K(P_X, x^*) \geq S_K(P, x^*)$, and therefore x^* is not a possible cowinner. ■

Therefore, computing possible winners for K -approval with respect to the addition of *one* candidate can be done in polynomial time.

4.3. 2-approval, any number of new candidates

Let App_2 denote the 2-approval rule. Here, we assume that $|Y| \geq 2$ since our problem for K -approval with one new candidate has been shown polynomially solvable in the previous subsection. For any profile P and any alternative x' , we simply write $s(P, x')$ for the score of x' in P under App_2 (that is $s(P, x') = S_2(P, x')$, i.e. the number of times that x' is ranked within top two positions in P).

The algorithm. Let $P_X = P_1 \cup P_2 \cup P_3$, where P_1 consists of the votes in which x^* is *not* ranked within top two positions, P_2 consists of the votes in which x^* is ranked in the top position, and P_3 consists of the votes in which x^* is ranked in the second position. For any alternative x' , we let $\text{HP}(P_X, x')$ denote the set of votes in P_X , where x' is ranked in the second positions and x^* is *not* ranked in the first

positions (HP stands for “high priority”). For any alternative x' , we let $\text{MP}(P_X, x')$ denote the set of votes in P_X , where x^* is ranked in the top positions and x' is ranked in the second positions (MP stands for “medium priority”). It follows that $P_1 = \cup_{x' \neq x^*} \text{HP}(P_X, x')$ and $P_2 = \cup_{x' \neq x^*} \text{MP}(P_X, x')$. For any alternative x' , we let $\text{LP}(P_X, x')$ denote the set of votes in P_X , where x' is ranked in the top positions and there exists $x'' \neq x^*$ that is ranked in the second positions (LP stands for “low priority”). We first make the following observation on the $(X \cup Y)$ -profiles P' extending P_X (or simply extensions) for which x^* is a co-winner.

Proposition 3 *If there exists an extension P' of P_X such that $x^* \in \text{App}_2(P')$, then there exists an extension P^* of P_X under which x^* is a co-winner, and P^* satisfies the following conditions.*

1. *For any $V \in P_X$, if x^* is ranked within top two positions in V , then x^* is also ranked within top two positions in V^* .*
2. *For any $V^* \in P^*$, it cannot be the case that the first-ranked alternative is not in Y but the second-ranked alternative is in Y .*
3. *For any $x' \in X \setminus \{x^*\}$ and any $V \in \text{MP}(P_X, x') \cup \text{LP}(P_X, x')$, if x' is not ranked within top two positions in V^* (the extension of V in P^*), then for any $W \in \text{HP}(P_X, x')$, x' is not ranked within top two positions in W^* .*

Proof: (1) This is because if in the extension V' , x^* is not in the top two positions, then we simply move all of the alternatives in Y ranked higher than x^* to the bottom positions. Let V^* denote the vote obtained in this way. By replacing V' with V^* , the score of x^* is increased by 1, and the score of any other alternative is increased no more than 1, which means that x^* is still a co-winner.

(2) If there exists $V' \in P'$ where $x' \in X$ is ranked in the first position and $y \in Y$ is ranked in the second position, then we simply obtain V^* by switching y and x' .

(3) states that for any alternative x' , whenever we want to reduce its score, we should first try to reduce the score by putting a new alternative $y \in Y$ in the top of some vote in $V \in \text{HP}(P_X, x')$. This is because by putting y at the top of some vote in $\text{HP}(P_X, x')$, we may use only one extra alternative $y' \in Y$ to reduce the score of the alternative that is ranked at the top of V by 1. Formally, suppose there exist $V_1 \in \text{HP}(P_X, x')$, $V_2 \in \text{MP}(P_X, x') \cup \text{LP}(P_X, x')$ such that x' is within top two positions of V_1' (the extension of V_1) but not within top two positions of V_2' (the extension of V_2). Let $y \in Y$ be any alternative that is ranked within the top two positions in V_2' . We let V_2^* denote the vote obtained from V_2' by moving y to the

bottom, and let V_1^* denote the vote obtained from V_1' by moving y to the top position. Next, we replace V_1' and V_2' by V_1^* and V_2^* , respectively. It follows that the score of any alternative does not change, which means that x^* is still a co-winner. Let us mention an important point. It can happen that $V_1' \in \text{LP}(P_X, x'')$ where x'' does not satisfy statement (3) for the moment. Formally, x'' and x' are ranked in the first and the second positions respectively for voter V_1' . After the change, for voter V_1^* , x'' is ranked in the second position. So, V_1^* will never change in the future. In conclusion, we can repeat this procedure until statement (3) is satisfied for every $x' \in X \setminus \{x^*\}$. Since after each iteration at least one additional voter will never be modified in the future, this procedure ends in $O(|P_X|)$ times. ■

Proposition 3 simply tells us that when looking for an extension that makes x^* a co-winner, it suffices to restrict our attention to the extensions that satisfy the conditions given in the proposition. Moreover, using (1) of Proposition 3, we deduce that $s(P^*, x^*) = s(P_X, x^*)$. Hence, for voters $V \in P_3$ (the voters in which x^* is ranked in the second position), we can assume that the new alternatives of Y are put in bottom positions in P^* . Let p_1, \dots, p_{m-1} denote the score differences between x_1, \dots, x_{m-1} and x^* . Without loss of generality, let $p_i \geq 0$ for any $i \leq m - 1$ (otherwise we only focus on those alternatives x_i 's with $p_i \geq 0$). Our objective is to reduce all score differences to 0, while keeping the score of any new alternative no more than $s(P_X, x^*)$.

Next, we present an outline of our algorithm. As discussed above, the bottom line is that when trying to reduce p_i , we first try to use the votes in $\text{HP}(P_X, x_i)$, then $\text{MP}(P_X, x_i)$, and finally $\text{LP}(P_X, x_i)$. This is because by putting some alternatives in Y in the top positions in the votes of $\text{HP}(P_X, x_i)$, it not only reduces p_i by 1, but also creates an opportunity that we can “pay” one extra alternative from Y to reduce p_j by 1, where x_j is the alternative ranked in the top of this vote. For votes in $\text{MP}(P_X, x_i)$, we can only reduce p_i by 1 without any other benefit. For votes in $\text{LP}(P_X, x_i)$ we will have to use 2 alternatives from Y to bring down p_i by 1 (however, if we already put some $y \in Y$ in the first position in order to reduce p_j , where x_j is the alternative that is ranked in the second position in the original vote, then we only need to pay one extra alternative in Y to reduce p_i). Therefore, the major hardness is to find the most efficient way to choose the votes in $\text{HP}(P_X, x_i)$ to reduce p_i , when $p_i \leq |\text{HP}(P_X, x_i)|$. We will reduce this to a max-flow instance to solve it. The outline of the algorithm is as follows.

Outline of the algorithm.

1. If for some alternative x_i , $|\text{HP}(P_X, x_i)| \leq p_i$, then by item (3) of Proposition 3, for each vote in $V \in \text{HP}(P_X, x_i)$, we put one alternative from Y in the top position of V (at this point we have not specified how to choose the alternatives in Y , let us put a special token e to represent an alternative in Y —later we will specify how to choose the alternatives in Y appropriately). Update the votes, denoted by V'_1, \dots, V'_n , respectively. We note that some of them rank alternatives from Y and some of them might not. Update the HP, MP, and LP sets, now denoted by HP' , MP' and LP' , respectively. (For any alternative x' and any vote $V \in P_X$, in the extension of V , denoted by V' , if the first-ranked alternative is in Y and the second-ranked alternative is $x' \neq x^*$, then we let $V' \in \text{MP}'(P_X, x')$.) Update p_1, \dots, p_{m-1} , denoted by p'_1, \dots, p'_{m-1} .
2. Now, we can partition the set of alternatives in $X \setminus \{x^*\}$ into two parts X_1 and X_2 in the following way: let X_2 be the set of alternatives x_i with $|\text{HP}(P_X, x_i)| \leq p_i$ (that is, X_2 is the set of alternatives that we mentioned in Step (1)); let $X_1 = X \setminus (X_2 \cup \{x^*\})$. For any $x_i \in X_2$, we now put alternatives in Y in the top positions in $\text{MP}'(P_X, x_i)$ in as many votes as possible, but no more than p'_i votes. This step is optimal, because for $x_i \in X_2$, p'_i is reduced by 1 by only using one of the alternatives in Y (we recall that for any $x_i \in X_2$, $\text{HP}'(P_X, x_i) = \emptyset$ after (1)). If $p'_i \leq |\text{MP}'(P_X, x_i)|$ for some i , then we are done with x_i ; in this case, we delete x_i from X_2 . Update the HP' , MP' , and LP' sets, denoted by HP^* , MP^* and LP^* , respectively. Update p'_1, \dots, p'_{m-1} , denoted by p^*_1, \dots, p^*_{m-1} .
3. Suppose up to now, we have used the alternatives in Y for T times (T could be larger than $|Y|$). If $|Y| \times s(P_X, x^*) < T$, we can terminate the algorithm since in this case, x^* cannot be a co-winner. Actually, at least one alternative y of Y receives $T/|Y|$ points. Thus, since $s(P_X, x^*) < T/|Y|$, y has a higher score than x^* . Using Proposition 3, we deduce that x^* cannot be a co-winner.
4. We note that for each $x_i \in X_2$, the only way to reduce p^*_i is to put two alternatives of Y in the top two positions in a vote V in $\text{LP}^*(P_X, x_i)$, because in the above steps we have used up all the votes in $\text{HP}^*(P_X, x_i)$ and $\text{MP}^*(P_X, x_i)$ to reduce p_i . Now reducing p^*_i by one will cost us two alternatives in Y , but meanwhile, p^*_j is also reduced by one, where x_j is the alternative that is ranked in the second position in the V . We must have that $x_j \in X_1$. We note that $\bigcup_{x_i \in X_2} \text{LP}^*(P_X, x_i) \subseteq \bigcup_{x_j \in X_1} \text{HP}^*(P_X, x_j)$. Next, we show how to choose votes in $\text{LP}^*(P_X, x_i)$ for any $x_i \in X_2$ by solving the following integer max-flow instance (note that in case either X_1 or X_2 is empty, we just

assume that the flow has a null value).

Vertices: $\{s, t\} \cup X_1 \cup X_2 \cup \bigcup_{x_i \in X_2} \text{LP}^*(P_X, x_i)$.

Edges: for any $x_j \in X_1$, we have a directed edge (s, x_j) with capacity p_j^* ; for any $x_i \in X_2$, any $V \in \text{LP}^*(P_X, x_i)$, and any $x_j \in X_1$, we have an edge (V, x_i) with capacity 1, and if x_j is ranked in the second position in V , then we have an edge (x_j, V) with capacity 1; for any $x_i \in X_2$, we have an edge (x_i, t) with capacity p_i^* .

We claim that the problem has a solution if and only if the value of the max-flow from s to t in the above instance is at least $\sum_{k \leq m-1} p_k^* + \sum_{x_i \in X_2} p_i^* - (|Y| \times s(P_X, x^*) - T)$. Observe that the flow does not necessarily brings all p_i^* to 0, so it may eventually be required to pay one more new candidate per vote to push (cf. steps 2 and 3).

Proof: Suppose the above max-flow instance has a solution whose value is at least $\sum_{k \leq m-1} p_k^* + \sum_{x_i \in X_2} p_i^* - (|Y|s(P_X, x^*) - T)$. We next show how to construct a solution to the problem instance. Because the instance is integral, it must exist an integer solution. We arbitrarily choose one integer solution f , which assigns each edge an integer that represents the flow.

1. For any $x_i \in X_2$ and any $V \in \text{LP}^*(P_X, x_i)$, if there is a flow from x_i to x_j via V , then we obtain V^* from V by putting two alternatives from Y in the top positions (that is, both p_i^* and p_j^* are reduced by 1, which comes at the cost of using alternatives in Y twice). It is possible since $|Y| \geq 2$.
2. For any $x_i \in X_2$, if $f(x_i, t) < p_i^*$, then we arbitrarily choose $p_i^* - f(x_i, t)$ votes $V \in \text{LP}^*(P_X, x_i)$ such that V^* is not defined in (1), and obtain V^* by putting two alternatives from Y in the top two positions (again, we will specify how to choose the two alternatives from Y later). It is possible since $|Y| \geq 2$.
3. For any $x_j \in X_1$, if $f(s, x_j) < p_j^*$, then we arbitrarily choose $p_j^* - f(s, x_j)$ votes $V \in \text{HP}^*(P_X, x_j)$ such that V^* is not defined above (in (1) or (2)), and then we obtain V^* by putting exactly one alternative from Y in the top position of V . It is possible since by construction $|\text{HP}^*(P_X, x_j)| \geq p_j$ for $x_j \in X_1$.
4. For any V^* , if an alternative $y \in Y$ is not chosen to be put in the first (or first two) positions, then it is ranked in the bottom position.

Next, we specify how to choose the alternatives from Y to put in the top positions in the extensions. We recall that in steps (1) and (2) in the algorithm, we

have used the alternatives from Y for T times. For any $k \leq m - 1$, we have to use the alternatives from Y for p_k^* times, and for each of $x_i \in X_2$, to reduce p_i^* that is not covered by any flow $x_j \rightarrow V \rightarrow x_i$, we use one extra alternative of Y . Therefore, the total number of times that the alternatives of Y are ranked in the top two positions is no more than $\sum_{i \leq m-1} p_i^* + (\sum_{x_i \in X_2} p_i^* - \sum_{x_i \in X_2} f(x_i, t)) = \sum_{i \leq m-1} p_i^* + (\sum_{x_i \in X_2} p_i^* - F_{max})$, where F_{max} is the maximum flow from s to t . Because $F_{max} \geq \sum_{k \leq m-1} p_k^* + \sum_{x_i \in X_2} p_i^* - (|Y|s(P_X, x^*) - T)$, we have the following calculation.

$$\begin{aligned}
& \sum_{k \leq m-1} p_k^* + \sum_{x_i \in X_2} p_i^* - F_{max} \\
\leq & \sum_{k \leq m-1} p_k^* + \sum_{x_i \in X_2} p_i^* - \left(\sum_{k \leq m-1} p_k^* + \sum_{x_i \in X_2} p_i^* - (|Y|s(P_X, x^*) - T) \right) \\
= & |Y|s(P_X, x^*) - T
\end{aligned}$$

That is, we put alternatives from Y in the top two positions in the extension for no more than $|Y|s(P_X, x^*)$ times in the algorithm. Now, for each alternative of Y , we make $s(P_X, x^*)$ copies of it. Whenever in the above process we pick one alternative from Y to put in the top or the second position in the extension of a vote, we pick it with the cyclic order $y_1 \rightarrow y_2 \dots \rightarrow y_{|Y|} \rightarrow y_1$. It follows that x^* is a co-winner under P^* , since using step (3) of the algorithm, we know that $|Y|s(P_X, x^*) - T \geq 0$.

Next, we show that if there exists a solution to the problem instance, then the value of a max-flow is at least $\sum_{k \leq m-1} p_k^* + \sum_{x_i \in X_2} p_i^* - (|Y|s(P_X, x^*) - T)$. Due to Proposition 3, any solution to the problem instance can be converted to a solution as in steps (1) through (3) in the outline of the algorithm. Now, for any $x_i \in X_2$, let l_i denote the number of votes $V \in LP^*(P_X, x_i)$ such that in its extension V^* , the top two positions are the alternatives of Y . We must have that $l_i \geq p_i^*$. For every $x_i \in X_2$, we arbitrarily choose $l_i - p_i^*$ such votes, and move the first ranked alternative to the bottom position. For any $x_j \in X_1$, let l_j denote the number of votes $V \in HP^*(P_X, x_j) \cup MP^*(P_X, x_j)$ such that in its extension V^* , an alternative from Y is ranked in the top position. We must have that $l_j \geq p_j^*$. For every $x_j \in X_1$, we arbitrarily choose $l_j - p_j^*$ such votes, and move the first ranked alternative to the bottom position.

Now, let there be a flow from x_j to x_i via V^* if $V \in LP^*(P_X, x_i)$ and the top two positions in V^* are both in Y . This defines a flow whose value is at least $\sum_{x_i \in X_2} p_i^* - \sum_{x_j \in X_1} (l_j - p_j^*)$. Because the score of any alternative of Y is no more than $s(P_X, x^*)$, we have that $|Y|s(P_X, x^*) - T \geq \sum_{k \leq m-1} l_k \geq \sum_{x_i \in X_2} p_i^* +$

$\sum_{x_j \in X_1} l_j$, or equivalently, $-\sum_{x_j \in X_1} l_j \geq \sum_{x_i \in X_2} p_i^* - (|Y|s(P_X, x^*) - T)$.
Hence, we get:

$$\begin{aligned} F_{max} &\geq \sum_{x_i \in X_2} p_i^* - \sum_{x_j \in X_1} (l_j - p_j^*) \\ &= \sum_{k \leq m-1} p_k^* - \sum_{x_j \in X_1} l_j \\ &\geq \sum_{k \leq m-1} p_k^* + \sum_{x_i \in X_2} p_i^* - (|Y|s(P_X, x^*) - T) \end{aligned}$$

■

Example 1 Let $X = \{x^*, x_1, \dots, x_6\}$ and consider the following profile P consisting of 19 votes (we only mention the first two candidates in each vote):

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}	v_{17}	v_{18}	v_{19}
x^*	x_1	x_2	x_3	x_1	x_1	x_1	x_2	x_2	x_2	x_2	x_2	x_3	x_3	x_3	x_3	x_3	x_3	x_4
x_1	x^*	x^*	x^*	x_4	x_4	x_5	x_1	x_3	x_4	x_5	x_5	x_1	x_2	x_4	x_4	x_5	x_6	x_6
\vdots																		

So far we leave the number of new candidates unspecified.

We have $P_2 = \{v_1\}$, $P_3 = \{v_2, v_3, v_4\}$ and $P_1 = \{v_5, \dots, v_{19}\}$.

The initial scores are $S_2^P(x^*) = 4$, $S_2^P(x_1) = 7$, $S_2^P(x_2) = 7$, $S_2^P(x_3) = 8$, $S_2^P(x_4) = 6$ and $S_2^P(x_5) = 4$ and $S_2^P(x_6) = 2$. The candidates to catch up for x^* are x_1 , x_2 , x_3 and x_4 , with the score differences $p_1 = 3$, $p_2 = 3$, $p_3 = 4$ and $p_4 = 2$. This is summarized together with the priority classification in the following table:

	HP	MP	LP	p_i
x_1	v_8, v_{13}	v_1	v_5, v_6, v_7	3
x_2	v_{14}		$v_8, v_9, v_{10}, v_{11}, v_{12}$	3
x_3	v_9		$v_{13}, v_{14}, v_{15}, v_{16}, v_{17}, v_{18}$	4
x_4	$v_5, v_6, v_{10}, v_{15}, v_{16}$		v_{19}	2

At step 1, we check if there are candidates x_i for which $|HP(P, x_i)| \leq p_i$. This is the case for x_1 , x_2 and x_3 , thus we put one new candidate on top of v_8 , v_9 , v_{13} and v_{14} . The updated table is as follows:

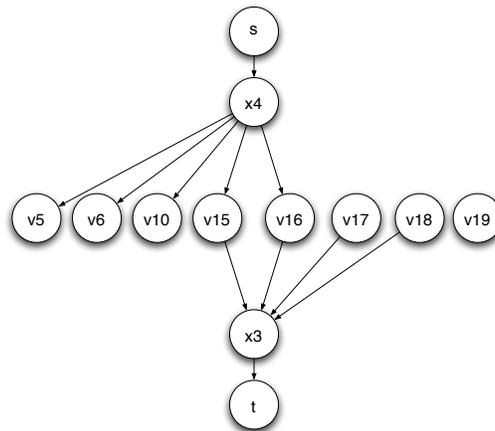
	<i>HP'</i>	<i>MP'</i>	<i>LP'</i>	p'_i
x_1		v_1	v_5, v_6, v_7	1
x_2		v'_8, v'_9	v_{10}, v_{11}, v_{12}	2
x_3		v'_{13}, v'_{14}	$v_{15}, v_{16}, v_{17}, v_{18}$	3
x_4	$v_5, v_6, v_{10}, v_{15}, v_{16}$		v_{19}	2

At step 2, $X_2 = \{x_1, x_2, x_3\}$ and $X_1 = \{x_4\}$ (we don't care about x_5 and x_6 for which nothing special has to be done). We push one new candidate in v_1, v'_8, v'_9, v'_{13} and v'_{14} , and we are done with x_1 and x_2 (since $p_1^* = 0$ and $p_2^* = 0$). The updated table is

	<i>HP*</i>	<i>MP*</i>	<i>LP*</i>	p_i
x_3			$v_{15}, v_{16}, v_{17}, v_{18}$	1
x_4	$v_5, v_6, v_{10}, v_{15}, v_{16}$		v_{19}	2

So far we have used the new candidates 9 times, and $S_2^P(x^*) = 4$, therefore if we have less than three new candidates we stop (c is not a possible cowinner) otherwise we continue. Now the situation is as follows and we have to solve the corresponding maxflow problem (we omit the value of edges when it equals 1).

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}	v_{17}	v_{18}	v_{19}
•	x_1	x_2	x_3	x_1	x_1	x_1	•	•	x_2	x_2	x_2	•	•	x_3	x_3	x_3	x_3	x_4
c	c	c	c	x_4	x_4	x_5	•	•	x_4	x_5	x_5	•	•	x_4	x_4	x_5	x_6	x_6
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮



The maximum flow has value 1 and is taken for instance by having a flow 1 for instance through the edges $s \rightarrow x_4$, $x_4 \rightarrow v_{16}$, $v_{16} \rightarrow x_3$, $x_3 \rightarrow t$ (going through v_{15} was an equally good option). Therefore we push two new candidates in v_{16} , which has the effect of making the score of x_3 and x_4 decrease by 1 each. We still have to make the score of x_4 decrease by 1, and for this we must push one new candidate in any of the votes v_5, v_6, v_{10}, v_{15} (say v_5). In total we will have used the new candidates 12 times, therefore, c is a possible co-winner if and only if the number of new candidates is at least 3. A possible extension (with 3 new candidates) is as follows:

v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}	v_{17}	v_{18}	v_{19}
\mathbf{y}_1	x_1	x_2	x_3	\mathbf{y}_2	x_1	x_1	\mathbf{y}_1	\mathbf{y}_2	x_2	x_2	x_2	\mathbf{y}_1	\mathbf{y}_2	x_3	\mathbf{y}_1	x_3	x_3	x_4
x^*	x^*	x^*	x^*	x_1	x_4	x_5	\mathbf{y}_3	\mathbf{y}_3	x_4	x_5	x_5	\mathbf{y}_3	\mathbf{y}_3	x_4	\mathbf{y}_2	x_5	x_6	x_6
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots

4.4. K -approval, two new candidates

Let $X = \{x^*\} \cup \{x_1, \dots, x_m\}$ be the set of (initial) candidates, x^* being the candidate that we want to make a co-winner, $\{y, z\}$ the new two candidates, and $P_X = \langle V_1, \dots, V_n \rangle$ the initial profile, where each V_i is a sequence of K candidates in X . We first introduce the following notation: for any $x \in X$, $U^{P_X}(x)$ is the number of votes in P_X whose candidates ranked $K - 1$ and K are respectively x and x^* , and $T^{P_X}(x) = S_{K-2}^{P_X}(x) + U^{P_X}(x)$. (Recall that $S_{K-2}^{P_X}(x)$ is the number of voters in P_X who rank x in the first $K - 2$ positions.)

The following Lemma gives us a necessary condition for a candidate x^* to be a possible co-winner.

Lemma 1 *For any $x \in X$, there exists an extension Q of P_X by adding two candidates such that $S_K^Q(x) \leq S_K^Q(x^*)$ if and only if $T^{P_X}(x) \leq S_K^{P_X}(x^*)$.*

Proof: Assume $T^{P_X}(x) > S_K^{P_X}(x^*)$, and let Q be a completion of P by adding two candidates. Let us partition P_X into P_1, P_2 and P_3 , such that: every vote in P_1 is such that the candidates ranked $K - 1$ and K are respectively x and x^* ; P_2 contains all votes ranking x in the first $K - 2$ positions; and P_3 contains all other votes in P_X . Let Q_1, Q_2 and Q_3 be the corresponding votes in Q , and let α be the number of votes in Q_1 where the two new candidates have been pushed towards the first K positions, thus eliminating both x and x^* from the K first positions; clearly, we have $S_K^{Q_1}(x) = S_K^{P_1}(x) - \alpha = U^{P_X}(x) - \alpha$ and $S_K^{Q_1}(x^*) \leq S_K^{P_1}(x^*) - \alpha$ (the inequality can be strict, in case there are some votes in Q_1 where only one new candidate was pushed towards the first K positions). Now, regardless of

the position of the two new candidates, we have $S_K^{Q_2}(x) = S_{K-2}^{P_2}(x)$. We get $S_K^Q(x) = S_K^{Q_1}(x) + S_K^{Q_2}(x) + S_K^{Q_3}(x) \geq U^{P_X}(x) - \alpha + S_{K-2}^{P_2}(x) = T^{P_X}(x) - \alpha$, whereas $S_Q(x^*) \leq S_K^{P_X}(x^*) - \alpha$. The initial assumption $T^{P_X}(x) > S_K^{P_X}(x^*)$ implies $S_K^Q(x) > S_K^{P_X}(x^*) - \alpha$, therefore $S_K^Q(x) > S_K^Q(x^*)$.

Conversely, assume $T^{P_X}(x) \leq S_K^{P_X}(x^*)$, and let us build Q as follows: we introduce one new candidate on top of each vote of P_X that ranks x in position K , and two new candidate on top of each vote of P_X that ranks x in position $K - 1$ and $x' \neq x^*$ in position K . It is easy to check that $S_K^Q(x^*) = S_K^{P_X}(x^*)$. Now, the only votes of Q where x remains among the first K positions are those of Q_1 and of Q_2 , therefore $S_K^Q(x) = T^{P_X}(x) \leq S_K^{P_X}(x^*) = S_K^Q(x^*)$. ■

Proposition 4 *Deciding whether x^* is a possible cowinner for K -approval with respect to the addition of 2 new candidates is in \mathbf{P} .*

Proof: A consequence of Lemma 1 is that if $T^{P_X}(x) > S_K^{P_X}(x^*)$, then x^* cannot be a possible cowinner in P_X under 2-approval with 2 new candidates; and obviously, checking whether $T^{P_X}(x) > S_K^{P_X}(x^*)$ holds for some x can be done in polynomial time. Therefore, *from now on, we assume that $T^{P_X}(x) \leq S_K^{P_X}(x^*)$ holds for every $x \in X$ — assuming this will not change the complexity of the problem.*

We are now giving a polynomial reduction from the possible winner problem for K -approval and 2 new candidates to the possible winner problem for 2-approval and 2 new candidates, which we already know to be polynomial. The profile P_X is translated into the following profile R :

- the set of candidates is $X \cup \{x^*\} \cup \{z_j, 1 \leq j \leq \sum_{x \in X \cup \{x^*\}} S_{K-2}^{P_X}(x)\} \cup \{z'_j, 1 \leq j \leq S_{K-2}^{P_X}(x^*)\}$, where all z_j and z'_j are fresh candidates;
- for every vote V_i in P_X , we have in R a vote W_i including the candidates ranked in positions $K - 1$ and K of V_i , and then the remaining candidates in any order. We denote by R_1 the resulting set of votes.
- for every $x \in X$, we have $S_{K-2}^{P_X}(x)$ votes xz_j , and then the remaining candidates in any order. We denote by R_2 be the resulting set of votes.
- similarly, we have $S_{K-2}^{P_X}(x^*)$ votes $z'_j x^*$, and then the remaining candidates in any order. We denote by R_3 be the resulting set of votes.

We note that if $x \in X \cup \{x^*\}$ then $S_K^{P_X}(x) = S_2^R(x)$, and for every fresh candidate z , $S_2^R(z) = 1$. Without loss of generality we assume $S_2^{P_X}(c) \geq 1$ (otherwise we know for sure that x^* cannot be a possible cowinner).

Suppose that x^* is a possible cowinner for K -approval with 2 new candidates in P_X and let $P' = \langle V'_1, \dots, V'_n \rangle$ be an extension of P_X with two new candidates where x is a cowinner. Then it is possible to use these two new candidates at most $2S_2^{P_X}(x^*)$ each, where $S_2^{P_X}(x^*)$ is the 2-approval score of x^* in P_X , to “push” candidates out of some votes in P_X . Let us use these two new candidates in the same way in R : every time a new candidate is used for pushing in V_i , it is also used for pushing in W_i . Let R' be the resulting profile. All candidates in $X \cup \{x^*\}$ have the same scores in P_X and in R , they also will have the same scores in P' and R' ; as for the fresh candidates z , $S_2^{R'}(z) = 1 \leq S_2^{R'}(x^*)$; therefore, x^* is a cowinner in R' and a possible cowinner for 2-approval with 2 new candidates in R .

Conversely, suppose that x^* is a possible cowinner for 2-approval with 2 new candidates in R , and let R' be a completion of R where x is a cowinner for 2-approval. Let us write $R' = R'_1 \cup R'_2$, where R'_1 (resp. R'_2) consists in the completions of the votes in R_1 (resp. R_2). Let $x \in X$ and assume that the following conditions hold:

1. in R'_1 , x appears only in votes of the form xx^* (all other votes in R_1 including x have been “pushed”);
2. $S_2^{R_2}(x) - S_2^{R'_2}(x) = \alpha > 0$: α votes in R_2 including x (those of the form xz_j) have been pushed.

From 1 we get $S_2^{R_2}(x) \leq U^{P_X}(x)$. Note also that by construction of R we have $S_2^{R_2}(x) = S_{K-2}^{P_X}(x)$. Now, from 1 and 2 we get $S_2^{R'}(x) = S_2^{R'_1}(x) + S_2^{R'_2}(x) = S_2^{R'_1}(x) + S_2^{R_2}(x) - \alpha \leq U^{P_X}(x) + S_{K-2}^{P_X}(x) - \alpha = T^{P_X}(x) - \alpha \leq S^{P_X}(c) - \alpha$ using the initial assumption. Consider now the profile R'' obtained from R' by replacing the α votes in R'_2 originating from a vote in R_2 of the form xz_i and that have been pushed, by the initial vote xz_i ; then $S_2^{R''}(x) = S_2^{R'}(x) + \alpha$, the scores of all other candidates in X being unchanged, the score of x^* being unchanged as well, and the scores of the two new candidates being lower than their score in R' : therefore, x^* is still a co-winner in R'' . Applying this iteratively for all $x \in X$, we end up with a completion R'' with no push in votes of R_2 , thus we conclude that *if x^* is a possible cowinner in R , then there exists an extension R'' of R where $R''_2 = R_2$ and x^* is a cowinner in R''_2* . Now, consider the following extension P'' or P_X : replace every vote $V = (v_1, \dots, v_{K-2}, v_{K-1}, v_K)$ of P_X by

$V'' = (v_1, \dots, v_{K-2}, v''_{K-1}, v''_K)$ where (v''_{K-1}, v''_K) is the vote in R'' corresponding to the completion of (v_{K-1}, v_K) . The scores of all candidates in $X \cup \{x^*\}$ are the same in P'' and in R'' , and the scores of the fresh candidates in P'' is $1 \leq S_{P''}(x^*)$, therefore x^* is a cowinner in P'' and a possible cowinner in P_X .

From this we conclude that deciding whether x^* is a possible cowinner for K -approval with respect to the addition of two candidates can be polynomially reduced to a problem of a deciding whether x^* is a possible cowinner for 2-approval, which we know is in \mathbf{P} . ■

4.5. 3-approval, 3 new candidates

Proposition 5 *Deciding if x^* is a possible winner for 3-approval with respect to the addition of 3 new candidates, is an NP-complete problem.*

Proof: This problem is clearly in NP. The proof is based on a reduction from the 3-dimensional matching problem, denoted by 3-DM. An instance of 3-DM consists of a subset $\mathcal{C} = \{e_1, \dots, e_m\} \subseteq X \times Y \times Z$ of triples, where X, Y, Z are 3 pairwise disjoint sets of size n with $X = \{x_1, \dots, x_n\}$, $Y = \{y_1, \dots, y_n\}$ and $Z = \{z_1, \dots, z_n\}$. A matching is a subset $M \subseteq \mathcal{C}$ such that no two elements in M agree in any coordinate, and the purpose of 3-DM is to answer the question: does there exist a perfect matching M on \mathcal{C} , that is, a matching of size n ? This problem with the restriction where no element of $X \cup Y \cup Z$ occurs in more than 3 triples is known to be NP-complete (problem [SP1] page 221 in [12]).

Let $I = (\mathcal{C}, X \times Y \times Z)$ be an instance of 3-DM with $n \geq 3$. For $a \in X \cup Y \cup Z$, $d(a)$ denotes the number of occurrences of a in \mathcal{C} , that is the number of triples of \mathcal{C} which contain a ; we can assume that $\forall a \in X \cup Y \cup Z$, $d(a) \in \{2, 3\}$. From I , we build an instance of the voting problem as follows. The set C of candidates contains x^* , $C_1 = \{x'_i, y'_i, z'_i : 1 \leq i \leq n\}$ where x'_i, y'_i, z'_i correspond to elements of $X \cup Y \cup Z$ and a set C_2 of dummy candidates. The set V of voters contains $V_1 = \{v^e : e \in \mathcal{C}\}$ and a set V_2 of dummy voters. For each voter, we only indicate her three first candidates. Thus, the vote of v^e is (x'_i, y'_j, z'_k) where $e = (x_i, y_j, z_k) \in \mathcal{C}$. The preference of dummy voters are such that (i) the score of the candidates in C verifies $\forall c \in C_1$, $score_s(c, P_C) = n + 1$, $score_s(x^*, P_C) = n$ and $\forall c \in C_2$, $score_s(c, P_C) = 1$ and (ii) any voter of V_2 contains at most one candidate of $\{x'_i, y'_i, z'_i : 1 \leq i \leq n\}$ in positions up to 3, and if he contains one, then it is in top position.

Formally, the instance of the voting problem is described as follows: The set of voters is $V = V_1 \cup V_2$ where $V_1 = \{v^e : e \in \mathcal{C}\}$ and $V_2 = V_X \cup V_Y \cup V_Z \cup V_{x^*}$, the set of candidates is $C = C_1 \cup C_2$ where $C_1 = X' \cup Y' \cup Z'$. For the candidates in C , we have

- $C_1 = X' \cup Y' \cup Z'$ where $X' = \{x'_1, \dots, x'_n\}$, $Y' = \{y'_1, \dots, y'_n\}$ and $Z' = \{z'_1, \dots, z'_n\}$.
- $C_2 = \{x^*\} \cup \{x_i^* : i = 1 \leq i \leq 2n\} \cup \{x_i^j : 1 \leq i \leq n, 1 \leq j \leq 2(n - d(x_i) + 1)\} \cup \{y_i^j : 1 \leq i \leq n, 1 \leq j \leq 2(n - d(y_i) + 1)\} \cup \{z_i^j : 1 \leq i \leq n, 1 \leq j \leq 2(n - d(z_i) + 1)\}$.

Note that $n - d(x_i) + 1 \geq 1$ since $d(a) \leq 3 \leq n$.

For each voter $i \in V$, we only indicate her three first candidates (in the order of preference). The set of all C -votes \mathcal{P}_C is given by

- $V_X = \{v_{i,j}^X : 1 \leq i \leq n, 0 \leq j \leq (n - d(x_i))\}$. The vote of $v_{i,j}^X$ is $(x'_i, x_i^{2j+1}, x_i^{2j+2})$.
- $V_Y = \{v_{i,j}^Y : 1 \leq i \leq n, 0 \leq j \leq (n - d(y_i))\}$. The vote of $v_{i,j}^Y$ is $(y'_i, y_i^{2j+1}, y_i^{2j+2})$.
- $V_Z = \{v_{i,j}^Z : 1 \leq i \leq n, 0 \leq j \leq (n - d(z_i))\}$. The vote of $v_{i,j}^Z$ is $(z'_i, z_i^{2j+1}, z_i^{2j+2})$.
- $V_1 = \{v^e : e \in \mathcal{C}\}$. The vote of v^e is (x'_i, y'_j, z'_k) where $e = (x_i, y_j, z_k) \in \mathcal{C}$.
- $V_{x^*} = \{v_j^{x^*} : 0 \leq j \leq n - 1\}$. The vote of $v_j^{x^*}$ is $(x^*, x_{2j+1}^*, x_{2j+2}^*)$.

We claim that I admits a perfect matching $M \subseteq \mathcal{C}$ if and only if x^* becomes a possible winner by adding three new candidates y_i^* , $i = 1, 2, 3$.

Let y_i^* for $i = 1, 2, 3$ be the new candidates added ($y_i^* \notin C$). Since we cannot increase the score of x^* , we must decrease by one point the score of candidates of $X' \cup Y' \cup Z'$.

Let us focus on candidates in X' . In order to reduce the score of x'_i , we must modify the preference of voters V_1 or V_X . By construction, each such voter must put y_1^*, y_2^*, y_3^* in positions up to 3 (since in V_1 or from (ii), candidates of X' are put in top position when they appear in position up to 3) and then, the score of y_i^* increases by 1 at each time. Since there are n candidates in X' , we deduce

$score_{\vec{s}}(y_i^*, P) \geq n$ for every $i = 1, 2, 3$. On the other hand, if x^* becomes a winner, $score_{\vec{s}}(y_i^*, P) \leq score_{\vec{s}}(x^*, P) \leq score_{\vec{s}}(x^*, P_C) = n$ from (i). Thus, $score_{\vec{s}}(y_i^*, P) = n$ for every $i = 1, 2, 3$ and there are exactly n voters V' which put y_1^*, y_2^*, y_3^* in position up to 3 (for the remaining voters of $V \setminus V'$, y_i^* is put in position at least 4 for every $i = 1, 2, 3$).

We affirm that $V' \subseteq V_1$. Otherwise, at least one voter of V_X put y_1^*, y_2^*, y_3^* in position up to 3. It remains at most $n - 1$ voters of V' to decrease by 1 the score of candidates in Y' . It is impossible because $|Y'| = n$ and from (ii) (and by construction of V_1) each candidate of Y' appears at most once in position up to 3 for all voters. Finally, since the score of candidates in $Y' \cup Z'$ must also decrease by 1, we deduce that x^* is a possible winner iff $M = \{e \in \mathcal{C} : y_1^*, y_2^*, y_3^* \text{ are in positions up to 3 for voter } v_e\}$ is a perfect matching of \mathcal{C} . ■

4.6. General case

We finalize the study of the possible winner problem for K -approval with respect to candidate addition by showing the following:

Proposition 6 *Deciding whether a candidate is a possible winner for K -approval with respect to the addition of k new candidates in NP-complete for any (K, k) such that $K \geq 3$ and $k \geq 3$.*

Proof: We start by giving a polynomial reduction from the possible winner problem for 3-approval with $k \geq 3$ new candidates to 3-approval with $k + 1$ new candidates. Let $\langle X, P, x^* \rangle$ where $P = \langle V_1, \dots, V_n \rangle$ be an instance of the possible winner problem for 3-approval with k new candidates (which we now know to be NP-complete for $k = 3$). Consider the following instance Q of 3-approval with $k + 1$ new candidates:

- the set of candidates is $X' = X \cup \{z\} \cup \{t_i^1, t_i^2 \mid 1 \leq i \leq 2S_3^P(x^*)\}$;
- there are $n + 2S_K^P(x^*)$ votes:
 - for every vote V in P we have a vote W in Q whose first three candidates are the same as in V and in the same order, and the other candidates are in an arbitrary order.

- for every $i = 1, \dots, 2S_3^P(x^*)$, we have a vote U_i in which the first 3 candidates are t_i^1, t_i^2, z , the remaining candidates being ranked arbitrarily.

Assume x^* is a possible cowinner for P (w.r.t. the addition of k new candidates) and let P' be an extension of P where x^* is a cowinner. Let us call $Y = \{y_1, \dots, y_k\}$ (resp. $Y' = \{y_1, \dots, y_{k+1}\}$) the new candidates for the instance $\langle X, P, x^* \rangle$ (resp. $\langle X', Q, x^* \rangle$). Consider the following extension Q' of Q : for every vote V' of P' we have a vote W' in Q' whose 3 first candidates are the same as in V (and the remaining ones in an arbitrary order); and for every vote U_i such that $i \leq S_3^P(x^*)$ we have a vote U'_i whose first 3 candidates are y_{k+1}, t_i^1, t_i^2 and for $S_3^P(x^*) + 1 \leq i \leq 2S_3^P(x^*)$, we have a vote U'_i whose first 3 candidates are t_i^1, t_i^2, z . We check that Q' is an extension of Q . The score of all candidates in $X \cup \{y_1, \dots, y_k\}$ are the same in P' and Q' , while the score of each t_i^1, t_i^2 is 1, the score of z and of y_{k+1} are $S_3^P(x^*)$; therefore x^* is a cowinner in Q' and a possible cowinner in Q .

Conversely, assume x^* is a possible cowinner in Q and let Q' be an extension of Q in which x^* is a cowinner. Without loss of generality, we assume that $S_3^{Q'}(x^*) = S_3^Q(x^*) = S_3^P(x^*)$, since under K -approval it is never beneficial to decrease the score of x^* to make it a possible cowinner. We have $S_3^{Q'}(z) \leq S_3^{Q'}(x^*) = S_3^P(x^*)$ and $S_3^Q(z) = 2S_3^P(x^*)$, therefore a new candidate must be put above z in at least $S_3^P(x^*)$ votes U_i . We now prove that we can assume that y_{k+1} is ranked within top three positions in exactly $S_3^P(x^*)$ votes in Q' . For the sake of contradiction, let us assume that y_{k+1} is not ranked within top three positions in $S_3^P(x^*)$ votes in Q' . Among all possible extension of Q (in which x^* becomes a cowinner), let us choose one Q' such that in the extension of each U_i , at most one y_j is ranked within top three positions, and y_{k+1} is ranked within top three positions in the extensions of the votes in $\mathcal{U} = \{U_i : i = 1, \dots, 2S_3^P(x^*)\}$ for the maximum number of times. Without loss of generality, we assume that there exists p with $p \leq k$, such that y_1, y_2, \dots, y_p are all alternatives that are ranked within top three positions in the extensions of the votes in \mathcal{U} . We also assume that y_{k+1} is ranked within top three positions in at least one of the extensions of the votes in \mathcal{U} (otherwise we exchange the name of y_p and y_{k+1}) and at most $S_3^P(x^*) - 1$ times in the extensions of the votes in \mathcal{U} (because if y_{k+1} is ranked $S_3^P(x^*)$ times within top three positions, then by putting y_1, y_2, \dots, y_p in the bottom positions in the extension of \mathcal{U} , we are done.)

Let us make some observations:

- (i) $p \leq 2$, and the score of y_{k+1} is $S_3^P(x^*)$ in Q' .

(ii) For each vote $V \notin \mathcal{U}$, if y_{k+1} is ranked within top three positions in V , then y_1 and y_2 (if $p = 2$) are also ranked within top three positions in V .

For (i). Suppose, for the sake of contradiction, that the score of y_{k+1} in Q' is at most $S_3^P(x^*) - 1$. Let U_i be the vote in whose extension y_1 is ranked within top three positions. Let U'_i denote the extension. In U'_i , by exchanging the positions of y_1 and y_{k+1} , we obtain a new extension in which x^* is a cowinner. However, in this new extension, the score of y_{k+1} in the extension of \mathcal{U} is increased by one, which contradicts the maximality of score of y_{k+1} in the extension of \mathcal{U} . Now, suppose on the contrary that $p \geq 3$. Consider a vote $V \notin \mathcal{U}$ where y_{k+1} is ranked within the top three positions (it is possible since the global score of y_{k+1} is $S_3^P(x^*)$ from (i) and the partial score of y_{k+1} is at most $S_3^P(x^*) - 1$ in the extension of \mathcal{U}). Then, in vote V at least one candidate among y_1, y_2, y_3 is not ranked within top three positions, say y_1 . In V , we exchange the positions of y_{k+1} and y_1 and in vote U_i where y_1 is ranked within top three positions, we also exchange the positions of y_1 and y_{k+1} . Then, we obtain a new extension in which x^* is a cowinner, and that contradicts the maximality of the score of y_{k+1} in the extension of \mathcal{U} .

For (ii). For the sake of contradiction, assume that there exists a vote $V \notin \mathcal{U}$ such that y_{k+1} is ranked within top three positions in V , and y_1 (resp., y_2) is not ranked within top three positions in V . Similar with the proof for (i), in vote V , we exchange the positions of y_{k+1} and y_1 (resp., y_2) and in the extension of U_i where y_1 (resp., y_2) is ranked within top three positions, we also exchange the positions of y_1 and y_{k+1} . It follows that there is a contradiction.

Since $k \geq 3$, there exists $y_3 \neq y_{k+1}$ ($y_3 \neq y_1, y_2$ if $p = 2$ and $y_3 \neq y_1$ if $p = 1$) such that in at least one vote $V' \notin \mathcal{U}$ where y_{k+1} is ranked within top three positions, y_3 is not ranked within top three positions (if $p = 2$, it is clear from (ii) and if $p = 1$ then from (ii), there is a vote $V \notin \mathcal{U}$, where y_{k+1} and y_1 are ranked within top three positions in V . Hence, at most one candidate among y_2, y_3 is ranked within top three positions in V). Also, we can assume that the score of y_3 in Q' is $S_3^P(x^*)$. Otherwise, by exchange the positions of y_1 and y_3 in V' we obtain a new extension Q'' satisfying the same hypothesis as Q' and such that (ii) does not hold, contradiction. Because the score of y_3 in Q' is $S_3^P(x^*)$, there exists a vote $V \notin \mathcal{U}$, such that y_3 is ranked within top three positions and y_1 is not ranked within top three positions (otherwise, the score of y_1 in Q' is at least $S_3^P(x^*) + 1$, $S_3^P(x^*)$ points come from the voters where y_3 is ranked within top three positions and one point come from the voters of \mathcal{U}). In V and V' , we exchange the positions of y_1 and y_3 . As previously, we also obtain a new extension Q'' satisfying the same

hypothesis as Q' and such that (ii) does not hold, contradiction. In conclusion, y_{k+1} is ranked among the first 3 positions in exactly $S_3^P(x^*)$ of votes in \mathcal{U} .

Then y_{k+1} cannot be ranked among the first 3 positions in any completion W'_i of a vote W_i , otherwise its score would be larger than the score of x^* and x^* would not be a possible cowinner. Therefore, in $\langle W'_1, \dots, W'_n \rangle$, the only new candidates appearing in the first 3 positions are y_1, \dots, y_k , and their score is at most $S_3^P(x^*)$; therefore x^* is a cowinner for 3-approval in $\langle W'_1, \dots, W'_n \rangle$, and a possible cowinner for 3-approval in profile P .

Second, we give a polynomial reduction from the possible winner problem for K -approval with k new candidates to $K + 1$ -approval with k new candidates. Let $\langle X, P, x^* \rangle$ where $P = \langle V_1, \dots, V_n \rangle$ be an instance of the possible winner problem for K -approval with k new candidates. Consider the following instance R of $K + 1$ -approval with k new candidates:

- the set of candidates is $X' = X \cup \{t_i \mid 1 \leq i \leq n\}$;
- for every vote V_i in P we have a vote W_i in R whose top candidate is t_i and the candidates ranked in position 2 to $K + 1$ are the candidates ranked in positions 1 to K in V_i , the remaining candidates being ranked arbitrarily.

Assume x^* is a possible cowinner for P and let $P' = \langle V'_1, \dots, V'_n \rangle$ be an extension of P where x^* is a cowinner. Denote by y_1, \dots, y_k the new candidates. Consider the extension $R' = \langle W'_1, \dots, W'_n \rangle$ or R where W'_i ranks t_i first and then the candidates ranked in the first K positions in V'_i . For every $x \in X$ we have $S_{K+1}^{R'}(x) = S_K^{P'}(x)$; for every $i = 1, \dots, k$ we have $S_{K+1}^{R'}(y_i) = S_K^{P'}(y_i)$; and for every $j = 1, \dots, n$, we have $S_{K+1}^{R'}(t_j) = 1$. Therefore x^* is a possible cowinner in R' and a possible cowinner in R' .

Conversely, assume x^* is a possible cowinner in R and let $R' = \langle W'_1, \dots, W'_n \rangle$ be a completion of R in which it is a possible cowinner. Since none of the t_i threatens x^* , without loss of generality we assume t_i still appears in the first $K + 1$ positions of W'_i – otherwise, change W'_i by moving one new candidate from the top to the bottom of W'_i . Consider now the extension $P' = \langle V'_1, \dots, V'_n \rangle$ or P where V'_i is obtained from W'_i by removing all the t 's. Since exactly one z_i appears in the first $K + 1$ positions of W'_i , the K candidates approved in V'_i are exactly the $K + 1$ candidates approved in W'_i minus t_i . From this we conclude that for every $x \in X$ we have $S_{K+1}^{P'}(x) = S_K^{R'}(x)$ and for every $i = 1, \dots, k$ we have $S_{K+1}^{P'}(y_i) = S_K^{R'}(y_i)$. Therefore x^* is a possible cowinner in P' and a possible

cowinner in P .

Since deciding whether x^* is a possible winner for 3-approval with respect to the addition of 3 new candidates, using inductively the above reductions show that NP-hardness propagates to every $(K, k) \geq (3, 3)$. Hence, the problem is NP-complete as soon as $K \geq 3$ and $k \geq 3$. ■

We summarize the results obtained in this Section by the following table:

	one or two new candidates	at least 3 new candidates
plurality	P	P
2-approval	P	P
K -approval, $K \geq 3$	P	NP-complete

A final remark about *necessary* cowinners. Some candidates (other than the new candidates) can be necessary co-winners with K -approval. Specifically, any candidate x_i such that $S_{K-k}(P_X, x_i) = n$ is a necessary cowinner, since she is approved by all voters and there are not enough new candidates to push her (in at least one vote) from the set of approved candidates.

We end up this Section by saying a few words about the veto rule. We said earlier that the possible winner problem for the veto rule is polynomial whatever the number of candidates, which at first glance could seem at odds with the above table, since veto coincides with $p - 1$ -approval, *when the number of candidates is fixed to p* . However, the above NP-hardness results hold for an a-unbounded number of candidates and therefore cannot capture the veto rule.

5. Borda

Let us now consider the Borda rule (r_B). Characterizing possible Borda winners when adding candidates is easy due to the following lemma:

Lemma 2 *Let P_X be a X -profile and let $y_1 \dots y_k$ be k new candidates. Let r be a scoring rule defined by the vector $\langle s_1 \dots s_p \dots s_{p+k} \rangle$ such that $(s_i - s_{i+1}) \leq (s_{i+1} - s_{i+2})$ for all i . $x^* \in X$ is a possible co-winner for P_X w.r.t. the addition of k new candidates iff $x^* \in r(P)$ where P is the profile on $X \cup \{y_1 \dots y_k\}$ obtained from P_X by putting $y_1 \dots y_k$ right below x^* (in arbitrary order) in every vote of P_X .*

Proof: We show that it is never beneficial to put the new candidates anywhere but right below x in the new profile. Let P be an extension of P_X in which x^* is a cowinner, and assume there is a vote $v \in P_X$ and a new candidate y , such that, either (i) $y \succ_v x^*$ or (ii) there is a candidate x' such that $x^* \succ_v x' \succ_v y$.

If we are in case (i), let us move y right below x^* ; let v' the resulting vote, and P' the resulting profile. Obviously, $S_B^{P'}(y) \leq S_B^P(y)$ and $S_B^{P'}(x^*) \geq S_B^P(x^*)$, therefore $S_B^{P'}(x^*) \geq S_B^{P'}(y)$. For any candidate z such that $y \succ_v z \succ_v x^*$, let i be the rank of z in v and $j > i$ be the rank of x^* in v . Then $(S_B^{P'}(z) - S_B^{P'}(x^*)) - (S_B^P(z) - S_B^P(x^*)) = (s_{i-1} - s_{j-1}) - (s_i - s_j) = (s_{i-1} - s_i) - (s_{j-1} - s_j) \leq 0$, therefore $S_B^{P'}(x^*) \geq S_B^{P'}(z)$. The scores of all other candidates do not change, therefore x^* is still a cowinner in P' . We apply this process iteratively and in all votes, until (i) no longer holds, and we result in a profile Q in which x^* is a cowinner and (i) does not hold.

If (ii) holds in some vote v of Q , then again we move y right below x^* ; let v' the resulting vote and Q' the resulting profile. The score of y improves, but since y is Pareto-dominated by x^* in Q and therefore also in Q' , we have $S_B^{Q'}(x^*) \geq S_B^{Q'}(y)$. For any candidate z such that $x^* \succ_v z \succ_v y$, z moves down one position in Q' , therefore $S_B^{Q'}(z) \leq S_B^Q(z) \leq S_B^Q(x^*) = S_B^{Q'}(x^*)$. The scores of all other candidates do not change, therefore x^* is still a cowinner in Q' . We apply this process iteratively and in all votes, until (ii) no longer holds, and we result in a profile Q'' in which x^* is a cowinner and neither (i) nor (ii) holds.

We conclude that if x^* is a possible cowinner for a profile, then it is a cowinner in an extension of the profile where all new candidates have been placed right below x^* . ■

In words, the rules concerned by Lemma 2 are rules where the difference of scores between successive ranks can only become smaller or remain constant as we come closer to the highest ranks. This condition is satisfied by Borda (but not by plurality), by veto, and by rules such as “lexicographic veto”, where the scoring vector is $\langle M^p, M^p - M, M^p - M^2, \dots, M^p - M^{p-1}, 0 \rangle$ where $M > n$.

The following result then easily follows:

Proposition 7 *Let P_X be an n -voter profile on X , and $x \in X$. A candidate x^* is a possible cowinner for Borda with respect to the addition of k new candidates if and only if*

$$k \geq \max_{z \in X \setminus \{x^*\}} \frac{\max(0, \text{score}(z, P_X) - \text{score}(x^*, P_X))}{N_{P_X}(x^*, z)}$$

Proof: Consider the case of one new candidate y . By Lemma 2 we know that y has to be placed right below x^* in the $X \cup \{y\}$ -profile, hence $score(x^*, P_{X \cup \{y\}}) = score(x^*, P_X) + 1$, and similarly for the candidates above x^* . For all candidates ranked below x^* on the other hand, we have $score(x^*, P_{X \cup \{y\}}) = score(x^*, P_X)$. Checking whether x^* is a possible winner then amounts to check, for each other candidate z , whether there are enough votes where x^* is preferred to z to compensate the score difference with this candidate, *i.e.* $N_{P_X}(x^*, z) \geq \max(0, score(z, P_X) - score(x^*, P_X))$. This immediately generalizes to k new candidates. ■

This means that possible co-winners with respect to adding any number of new candidates can be computed in polynomial time in the case of the Borda rule (and more generally for rules satisfying the conditions of Lemma 2). Note that the general problem of computing possible winners for Borda is NP-complete [22], therefore, the restriction of the problem to candidate addition induces a complexity fall.

We give an example to illustrate this result. Suppose we have $X = \{a, b, c, d\}$, $n = 4$, and an initial profile $P_X = \langle bacd, bacd, bacd, dacb \rangle$. Let us denote by $\delta(x, z)$ the expression

$$\frac{\max(0, score(z, P_X) - score(x, P_X))}{N_{P_X}(x, z)}$$

Hence the values of $\delta(x, z)$ for any pair $x, z \in X$:

$$\delta(b, x) = 0 \text{ for any } x \neq b;$$

$$\delta(a, b) = \frac{9-8}{1} = 1; \delta(a, c) = \delta(a, d) = 0;$$

$$\delta(c, a) = \frac{8-4}{1} = +\infty; \delta(c, b) = \frac{9-4}{1} = 5; \delta(c, d) = 0;$$

$$\delta(d, a) = \frac{8-3}{1} = 5; \delta(d, b) = \frac{9-3}{1} = 6; \delta(d, c) = \frac{4-3}{1} = 1.$$

Therefore:

- with at least 1 new candidate, a is a possible cowinner;
- with at least 6 new candidates, d also becomes a possible cowinner;
- whatever the number of new candidates, c is never a possible cowinner.

Notice that c is never a possible cowinner although it has a higher Borda score than d in P_X .

6. Hardness with a single new candidate

Even though we have seen that the possible winner problem can be NP-hard for some scoring rules, NP-hardness required the addition of several new candidates. We now show that there exists a scoring rule for which the possible winner problem is NP-hard with respect to the addition of *one* new candidate.

The scoring rule we use is very simple: it allows each voter to approve exactly 3 candidates, and offers 3 different levels of approval (assigning respectively 3,2,1 points to the three preferred candidates). Let r_Δ be the scoring rule defined by the vector $\langle 3, 2, 1, 0, \dots, 0 \rangle$ with $p - 3$ 0's completing the vector.

Proposition 8 *Deciding if x^* is a possible co-winner for r_Δ with respect the addition of one candidate is NP-complete.*

Proof: This problem is clearly in NP. The proof is quite similar to that of Proposition 5. Let $I = (\mathcal{C}, X \times Y \times Z)$ be an instance of 3-DM with $n \geq 5$ and $\forall a \in X \cup Y \cup Z, d(a) \in \{2, 3\}$. From I , we build an instance of the voting problem as follows. The set \mathcal{C} of candidates contains x^* , $C_1 = \{x'_i, y'_i, z'_i : 1 \leq i \leq n\}$ where x'_i, y'_i, z'_i correspond to elements of $X \cup Y \cup Z$ and a set C_2 of dummy candidates. The set V of voters contains $V_1 = \{v^e : e \in \mathcal{C}\}$ and a set V_2 of dummy voters. For each voter $i \in V$, we only indicate the vote for the 3 first candidates. So, the vote (a_1, a_2, a_3) means that candidate a_i receives $4 - i$ points. Thus, the vote of v^e is (x'_i, y'_j, z'_k) where $e = (x_i, y_j, z_k) \in \mathcal{C}$. The preferences of dummy voters are such that (i) the score of the candidates in C verifies $\forall c \in C_1, score_{\vec{s}}(c, P_C) = 3n + 1, score_{\vec{s}}(x^*, P_C) = 3n$ and $\forall c \in C_2, score_{\vec{s}}(c, P_C) \leq 3$ and (ii) any voter of V_2 contains at most one candidate of $\{x'_i, y'_i, z'_i : 1 \leq i \leq n\}$ in positions up to 3, and if he contains one in second position, then x^* occurs in third position.

Formally, the instance of the voting problem is built as follows. The set of voters is $V = V_1 \cup V_2$ where $V_1 = \{v^e : e \in \mathcal{C}\}$ and $V_2 = V_X \cup V_Y \cup V_Z \cup V_{x^*}$, the set of candidates is $C = C_1 \cup C_2$ where $C_1 = X' \cup Y' \cup Z'$ with $X' = \{x'_1, \dots, x'_n\}$, $Y' = \{y'_1, \dots, y'_n\}$, $Z' = \{z'_1, \dots, z'_n\}$ and $C_2 = C_X \cup C_Y \cup C_Z \cup C_{x^*}$. These two sets are described in the following way:

- $C_X = \{x_i^j : 1 \leq i \leq n, 1 \leq j \leq 2(n - d(x_i))\}$.
- $C_Y = \{y_i^j : 1 \leq i \leq n, 1 \leq j \leq 2(3n - 2d(y_i) + 1)\}$.
- $C_Z = \{z_i^j : 1 \leq i \leq n, 1 \leq j \leq 2(3n - d(z_i) + 1)\}$.

- $C_{x^*} = \{x^*\} \cup \{x_i^* : i = 1 \leq i \leq 2n\}$.

For each voter $i \in V$, we only indicate the vote for the 3 first candidates. So, the vote (a_1, a_2, a_3) means that candidate a_i receive $4 - i$ points. The set of all C -votes \mathcal{P}_C is given by:

- $V_X = \{v_{i,j}^X : 1 \leq i \leq n, 0 \leq j \leq (n - d(x_i) - 2)\} \cup \{v_i^{X,j} : 1 \leq i \leq n, j = 1, 2\}$. The vote of $v_{i,j}^X$ is $(x'_i, x_i^{2j+1}, x_i^{2j+2})$. Note that $n - d(x_i) - 2 \geq 0$. The vote of $v_i^{X,j}$ is $(x_i^{2(n-d(x_i)-1)+j}, x'_i, x^*)$.
- $V_Y = \{v_{i,j}^Y : 1 \leq i \leq n, 0 \leq j \leq 3n - 2d(y_i)\}$. The vote of $v_{i,j}^Y$ is $(y_i^{2j+1}, y_i^{2j+2}, y'_i)$.
- $V_Z = \{v_{i,j}^Z : 1 \leq i \leq n, 0 \leq j \leq 3n - d(z_i)\}$. The vote of $v_{i,j}^Z$ is $(z_i^{2j+1}, z_i^{2j+2}, z'_i)$.
- $V_{x^*} = \{v_j^{x^*} : 0 \leq j \leq n - 1\}$. The vote of $v_j^{x^*}$ is $(v_{2j+1}^{x^*}, v_{2j+2}^{x^*}, x^*)$. Note that $n - 1 \geq 0$.
- $V_1 = \{v^e : e \in \mathcal{C}\}$. The vote of v^e is (x'_i, y'_j, z'_k) where $e = (x_i, y_j, z_k) \in \mathcal{C}$.

We claim that I admits a perfect matching $M \subseteq \mathcal{C}$ if and only if x^* becomes a possible winner by adding a new candidate y_1 .

Observe that the scores of the candidates in C verify:

- (i) $\forall c \in C_1, \text{score}_{\bar{s}}(c, P_C) = 3n + 1$.
- (ii) $\text{score}_{\bar{s}}(x^*, P_C) = 3n$.
- (iii) $\forall c \in C_2 \setminus \{x^*\}, \text{score}_{\bar{s}}(c, P_C) \leq 3$.

Actually, for (i) we get $\forall i \in \{1, \dots, n\}, \text{score}_{\bar{s}}(x'_i, P_C) = 3d(x_i) + 3(n - d(x_i) - 1) + 2 + 2 = 3n + 1$ (candidate x'_i receives 3, 3, 2 and 0 points respectively for voters $v^e, v_{i,j}^X, v_i^{X,j}$ and the remaining voters), $\text{score}_{\bar{s}}(y'_i, P_C) = 2d(y_i) + (3n - 2d(y_i) + 1) = 3n + 1$ (candidate y'_i receives 2, 1 and 0 points respectively for voters $v^e, v_{i,j}^Y$ and the remaining voters) and $\text{score}_{\bar{s}}(z'_i, P_C) = d(z_i) + (3n - d(y_i) + 1) = 3n + 1$ (candidate x'_i receives 1, 1 and 0 points respectively for voters $v^e, v_{i,j}^Z$ and the remaining voters). For (ii) we have $\text{score}_{\bar{s}}(x^*, P_C) = 2n + n = 3n$ (each voter $v_j^{x^*}$ and $v_i^{X,j}$ gives 1 point for candidate x^*). Finally, for (iii) we get $\forall c \in C_2 \setminus \{x^*\},$

$score_{\vec{s}}(c, P_C) \leq 3$ since each candidate of $C_2 \setminus \{x^*\}$ has a positive score for exactly one voter and the maximum score is 3.

Let y_1 be the new candidate. By construction of this scoring rule, we must decrease the score of candidates in C which dominate the score of x^* , that is the candidates of C_1 using (i) and (iii).

Let us focus on candidates of X' . In order to reduce to one unit the score of x'_i , we must modify the preference for at least one voter v^e or $v_{i,j}^X$ or $v_i^{X,j}$. If we modify it for some voter in $v_i^{X,j}$, then the score of x'_i (with respect to $v_i^{X,j}$) decreases by one iff the score of x^* (with respect to $v_i^{X,j}$) also decreases by one. In conclusion, we must modify the preference of x'_i for at least one voter v^e or $v_i^{X,j}$. By construction, each such voters must put y_1 in top position and then, the score of y_1 increases by 3 at each time. Since there are n candidates in X' , we deduce $score_{\vec{s}}(y_1, P) \geq 3n$; From above remark, we also get $score_{\vec{s}}(x^*, P) \leq score_{\vec{s}}(x^*, P_C) = 3n$. Thus for each $i \in \{1, \dots, n\}$, exactly one voter among those of v^e or $v_i^{X,j}$ must put candidate y^* in top position. Finally, if it is one voter $v_i^{X,j}$, then we deduce $score_{\vec{s}}(y_1, P) > 3n$ because the score of $Y' \cup Z'$ must also decrease, which is not possible since y_1 will become winner.

Following a line of reasoning similar to the one developed in the proof of Proposition 5, we conclude that for each $i \in \{1, \dots, n\}$, exactly one voter among those of v^e must put candidate y_1 in top position. Since the the score of $Y' \cup Z'$ must also decrease by 1, we deduce that x^* is a possible winner iff $M = \{e \in \mathcal{C} : y_1 \text{ is in top position for voter } v_e\}$ is a perfect matching of \mathcal{C} (for the remaining voters, y_1 is put in last position). ■

This rule is an example for which it is difficult to identify possible winners with a single missing candidate. Giving a characterization of those rules sharing this property is an open problem.

7. Related work

7.1. The possible winner problem

The possible winner problem was first introduced in [15]: given an incomplete profile $P = \langle V_1, \dots, V_n \rangle$ where each V_i is a partial order over the set of candidates X , x is a possible winner for P given a voting rule r if there exists a complete extension $P' = \langle V'_1, \dots, V'_n \rangle$ of P , where each V'_i is a linear order on X extending V_i , such that $r(P') = x$. The possible winner can of course be defined in a similar

way for a voting correspondence C , in which case we say that x is a possible cowinner if there exists an extension P' of P such that $x \in C(P')$. Clearly, the possible winner problem defined in this paper is a restriction of the general possible winner problem to the following set of incomplete profiles:

(Restr) *there exists a subset X' of candidates such that for every i , V_i is a linear order on X'*

As an immediate corollary, the complexity of possible winner with respect to candidate addition is at most as difficult as that of the general problem. This raises the question whether (Restr) leads to a complexity fall or not for the scoring rules we have considered here.

The possible (co)winner problem for scoring rules has received a deep attention in the last years. [22] have proven that the problem was NP-complete for the Borda rule (and more generally for scoring rules whose scoring vector contains four consecutive, equally decreasing values, followed by another strictly decreasing value). Then [4] went further by showing that NP-completeness holds more generally for all pure¹ scoring rules, except plurality, veto, and scoring rules whose vector s^n is of the form $s^n = (2, 1, \dots, 1, 0)$ from a given n^* on. The issue was finally closed by [3], who showed that the problem for $s^n = (2, 1, \dots, 1, 0)$ is NP-complete as well. These results compare to ours in the following way: all our NP-hardness strengthen the known NP-hardness results for the general possible winner problem, while our polynomiality results show a complexity fall induced by (Restr). For the sake of completeness, we also mention the complexity of the other prominent subproblem, namely unweighted coalitional manipulation (see e.g., [24]).

	general problem	candidate addition	manipulation
plurality and veto	P	P (Prop. 1)	P
Borda	NP-complete [22]	P (Prop. 7)	?
2-approval	NP-complete [4]	P (Prop. 4)	P
K -approval, $K \geq 3$	NP-complete [4]	NP-complete (Prop. 5)	P

Another interesting line of work is the the *parameterized* complexity of the possible winner problem for scoring rules, which has been investigated in [5].

¹A (family of) scoring rules $(r_n)_{n \geq 1}$ is pure if for any n , the scoring vector for $n+1$ candidates is obtained from the scoring vector for n candidates by inserting an additional score at an arbitrary position. All interesting families of scoring rules are pure; this is in particular the case for K -approval and Borda.

Among other results, they show that for all scoring rules, the problem is fixed-parameter tractable with respect to the number of candidates (in particular, when the number of candidates is bounded by a constant, the problem becomes polynomial). This polynomiality result clearly extends to the possible winner problem with respect to candidate addition, with some caution: the number of candidates here is the *total* number of candidates (the initial ones plus the new ones); this result has a practical impact in some situations mentioned in the introduction, such as finding a date for a meeting, where the number of candidates is typically low.

We end this subsection by mentioning other works on the possible winner problem and its variants and subproblems, that are less directly connected to our results. The possible winner problem has also been studied from the probabilistic point of view [1], where the aim is to count the number of extensions in which a given candidate is the winner. Such a probabilistic analysis is highly relevant in candidate-adding situations: given P_X , a number k of new candidates, and a prior probability distribution on votes, computing the probability that given candidate $x \in X$ will be the winner, or that one of the initial (resp. new) candidates will be the winner, is extremely interesting².

7.2. Control via adding candidates

The possible winners with respect to the addition of candidates is highly reminiscent of constructive control by the chair via adding candidates — this problem first appeared in [2] and was later studied in more depth for many voting rules, see *e.g.*, [14, 11]. However, even if a voting situation where new candidates are added looks similar to an instance of constructive control of an election by adding candidates, both problems differ significantly. In control via adding candidates, the input consists of a set of candidates X , a set of “spoiler” candidates Y , and a full profile $P_{X \cup Y}$: the chair knows how the voters would vote on the new candidates; the problem is to determine whether a given candidate x^* can be made a winner by adding at most $k \leq |Y|$ candidates from Y . In the possible winner problem with respect to candidate addition, $P_{X \cup Y}$ we have to take into account all possible ways for voters to rank the new candidates. Therefore, in spite of their significant differences, there is a straightforward connection between both problems: if an

²Note that if the voting rule is neutral, then although the prior probability that one of the k new candidates will be a cowinner under the impartial culture assumption is at least $\frac{|X|+k}{k}$, this is no longer the case when P_X is known: for instance, let us use plurality and consider the profile $P_X = \langle ab, ab, ab \rangle$, and let the number of new candidates be one. Then the probability that the new candidate is a cowinner in the completed profile is only $\frac{1}{27}$.

instance $\langle X, Y, P_{X \cup Y}, x^*, k \rangle$ of control via adding candidates is positive, then x^* is a possible winner with respect to the addition of k new candidates (the voting rule being the same in both problems).

Bartholdi et al. [2] noted that a voting rule is immune to control by adding candidates as soon as it satisfies the Weak Axiom of Revealed Preference (WARP), which requires that the winner among a set of candidates be the winner among every subset of candidates to which he belongs [19]; formally: *for any $Z \subseteq Y$, if $r(P_Y) \in Z$ then $r(P_Z) = r(P_Y)$* . This property can be used in a similar way for the possible winner problem with respect to candidate addition: if the voting rule satisfies WARP, then an initial candidate is a possible winner if and only if it is the winner for the current profile P_X . Unfortunately, this social-choice theoretic property is very strong, and only degenerated voting rules satisfy it.

7.3. Cloning

Finally, the possible winner problem via candidate addition is highly related to manipulation by candidate cloning. Independence of clones was first studied in [20], further studied in [16, 17], and a variant of this property was recently considered from the computational point of view in [9]. The main difference between x being a possible winner with respect to candidate addition and the existence of a candidate cloning strategy so that x or one of its clones becomes the winner, as in [9], is that candidate cloning requires a candidate and its clones to be contiguous in all votes. In other terms, whereas our problem considers the introduction of genuinely new candidates, cloning merely copies of existing ones.

The complexity of this problem is considered by Elkind et al. [9] for several voting rules. Although the proposed model allows for the possibility of having a bounded number of new clones (via a notion of cost), most of their results focus on the case of unboundedly many clones. Therefore, to be able to compare their results with ours, we should first say something about the variant of the possible winner problem with respect to candidate addition, *when the number of new candidates is not known beforehand and can be arbitrarily large*. The definitions of voting situations and possible winners are straightforward adaptations of Definitions 1 and 2: a voting situation is now a triple $\Sigma = \langle N, X, P_X \rangle$ and x^* is a possible co-winner with respect to Σ and r if there exists an integer k and a set Y of cardinality k such that there is a $X \cup Y$ -profile P extending P_X such that $x^* \text{ in } r(P)$. We now give a necessary and sufficient condition for a candidate to be a possible winner, for a class of scoring rules including the Borda rule.

Proposition 9 *Let S be a collection of scoring vectors (s^p) , $p \geq 1$ such that*

- for every p , (s_j^p) , $1 \leq j \leq p$ is strictly decreasing;
- for all $j, j' \in \mathbb{N}$, (1) $\lim_{p \rightarrow \infty} \frac{s_j^p - s_{j'}^p}{s_1} = 0$ and (2) $\lim_{p \rightarrow \infty} \frac{s_j^p - s_{q-j'}^p}{s_1} = 1$.

Then, x^* is a possible winner w.r.t. the addition of an unbounded number of new candidates if and only if it is undominated.

Proof: First suppose x^* is undominated. For any candidate $x_i \neq x^*$, define Δ_i as the difference between the score of x and the score of x_i , divided by s_1 , in the current vote. As in the construction of Lemma 1, put k new candidates right below x^* in every vote. As the value of k grows, for votes ranking candidate x^* below x_i , the value of Δ_i will tend towards 0 (by condition 1). Also, condition 2 ensures that for each vote ranking x^* above x_i , the value of Δ_i tends towards 1. Because x^* is undominated, such votes always exist for all candidate $x_i \neq x^*$. Therefore, $\lim_{k \rightarrow \infty} \Delta_i \geq 1$ and x will eventually become the winner as k grows. Now suppose x^* is dominated by some candidate x_i . Because the scores (s_j^p) , $1 \leq j \leq p$ are strictly decreasing, the score of x will always remain strictly below the score of x_i in the completion of the profile, hence x^* is not a possible co-winner. ■

Clearly, this large class of voting rules includes the Borda rule since it satisfies the conditions of proposition 9. However, this does not include the plurality rule and more generally K -approval rules which violate condition (1). Still, a very simple condition can be stated for K -approval: a candidate is a possible winner as soon as it is approved at least once.

Proposition 10 *When r is K -approval, x is a possible winner w.r.t. the addition of an unbounded number of new candidates if and only if $S_K(P_X, x) \geq 1$.*

Proof: The condition is obviously necessary. Suppose the condition holds on a given profile. We extend this profile by taking a set of new candidates y_{ij} where $1 \leq i \leq n$ and $1 \leq j \leq K$. Consider the i -th vote: if x is ranked in one the k^{th} position, put all new candidates at the bottom of the vote. Otherwise, introduce the new candidates $\{y_{i1}, \dots, y_{iK}\}$ at the top of the votes, and all other new candidates at the bottom. The score of the new candidates is at most 1, while that of $x_i \neq x$ is at most that of x (which is unchanged). ■

Note that for $K \geq 2$ this condition does *not* imply that the candidate is undominated (nor vice-versa). It does obviously when $K = 1$, *i.e.* for plurality.

Let us see now how the above results relate to those in [9]. We first note that in the case of the Borda rule we have the same condition. Indeed one sees intuitively that Lemma 2 tells us that for some voting rules (inc. Borda), introducing new candidates in a contiguous manner, as with cloning, is the best thing to do. For plurality, again the condition is similar in both cases. However, for K -approval as soon as $K > 1$, the problem becomes hard in the cloning setting whereas it is easy in our setting with an unbounded number of new candidates.

8. Conclusion

In this paper we have considered voting situations where new candidates may show up during the process. This problem increasingly occurs in our societies, as many votes now take place online (through dedicated platforms, or simply by email exchange) during an extended period of time.

We have identified the computational complexity of computing possible winners for some scoring rules. Some of them allow polynomial algorithms for the problem (*e.g.* plurality, 2-approval, Borda, veto) regardless of the (fixed) number of new candidates showing up. For the rules of the K -approval family, when $K \geq 3$, the problem remains polynomial only if the number of new candidates is ≤ 2 . Finally, we have exhibited a simple rule where the problem is hard for only one new candidate.

The results address the problem of making some designated candidate a co-winner, which is similar to the x being unique winner under the assumption of the most favourable tie-breaking. In the other extreme case (if we want x to be a strict winner, *i.e.* win regardless of the tie-breaking rule), the results are easily adapted: for instance, the inequalities in Prop. 1 and 7 become strict. For K -approval, the first condition of Prop. 2 becomes strict but the second one should now read $S_K(P_X, x) \geq \sum_{x_i \in X} \max(0, S_K(P_X, x_i) - S_K(P_X, x) + 1)$. As for veto, all other initial candidates need to be vetoed at least once. The hardness proofs can also be readily adapted to the unique winner setting. A more general treatment would require cumbersome expressions, and is also somewhat problematic since the identity of the new candidate is not known anyway (making it difficult to specify easily a tie-breaking rule on these candidates).

As for future work, a first direction to follow would consist in trying to obtain more general results for scoring rules, as do [4] for the general version of the possible winner problem. Extending the study to other families of voting rules, such as rules based on the majority graph, is also worth investigating.

Of course, identifying possible winners is not the end of the story. In practice, as mentioned earlier, one may for instance also be interested in a refinement of this notion: knowing how likely it is that a given candidate will win. Another interesting issue consists in designing elicitation protocols when the preferences about the ‘old’ candidates are already known. In this case, a trade-off occurs between the storage cost and communication cost, since keeping track of more information is likely to help to reduce the burden of elicitation.

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