Two New Measurements of Voting Power

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Abstract

The analysis of voting power in elections has been playing an important role in many traditional social choice problems, where the voting power of an agent is measured by whether the agent is *pivotal*. In this paper, we introduce two new extensions of such pivotality to measure agents' voting power in a given profile. The first extension, called *hierarchical pivotal sets*, captures the voting power for an agent to make other agents pivotal. The second extension, called *coalitional pivotal sets*, captures the voting power for a group of agents to change the winner by voting differently in collaboration.

We then focus on characterizing hierarchical pivotal sets. We prove that for any voting rule that satisfies *anonymity* and *unanimity*, and for any given profile, the union of the hierarchical pivotal sets are a sound and complete characterization of the non-redundant agents. We also show that if the voting rule does not satisfy anonymity, then this characterization might not be complete. Finally, we investigate algorithmic aspects of computing both types of pivotal sets.

Introduction

Voting has been used in multiagent systems as a popular way to aggregate agents' preferences over a set of alternatives. Recently, a burgeoning field *computational social choice* was formed to study the computational aspects of voting. In computational social choice, one central problem is to investigate the possibility of using computational complexity as a barrier against manipulation. Researchers have been interested in the computational complexity of computing whether a single agent or a coalition of agents have enough voting power to replace the winner with their favorite alternative by casting votes strategically in collaboration. See (Faliszewski, Hemaspaandra, and Hemaspaandra 2010) and (Faliszewski and Procaccia 2010) for nice recent surveys.

Looking back in the literature, the study of voting power has been favored in Political Science and Economics for a long time. It has been playing a central role in at least two other main research directions in addition to the study of manipulation. The first direction is the study of rational choice of voters, motivated by the "paradox of not voting", which dates back to Downs' seminal work (Downs 1957). The paradox states that when the number of voters is large, the voting power for a single voter to influence the outcome is negligible. Therefore, nobody should bother to vote, which sharply contradicts the much higher turnout in real-life elections. The paradox of not voting has influenced the study of voting in Political Science for more than half a century, and is still popular nowadays. Many research papers have been devoted to explaining the paradox from both theoretical and empirical sides, yet none of them has been successful so far. See (Feddersen 2004; Geys 2006) for recent surveys.

In both research directions mentioned above, a voter's voting power is determined by whether or not she is *pivotal*. More precisely, in a given profile, a voter is pivotal if and only if she can change the winner by casting a different vote, assuming that the other voters do not change their votes[1]. However, the mere "pivotal or not" measurement is often not discriminative enough. As it has been illustrated in the paradox of not voting, the set of pivotal voters is always too small or even empty when the number of voters is large. This argument is supported by some recent work on the probability that a coalition of voters have power to change the outcome (Procaccia and Rosenschein 2007; Xia and Conitzer 2008).

Our conceptual contributions. In this paper, we introduce two new ways to measure a voter's power in a given profile for a given voting rule. Both ways are extensions of the set of pivotal agents, but are much more discriminative. We believe that these extensions provides new angles of the voters' strategic behavior in the three traditional research directions mentioned above. The first extension, called *hierarchical pivotal sets*, captures the power for a voter to make other voters pivotal. Given a profile, the level-1 hierarchical set is composed of all pivotal voters; for any $k \geq 2$, the level-k hier-

[1] In the study of voting power, we do not consider the voter's incentive to cast a different vote.

archical set is composed of all voters who can change the level-$(k-1)$ hierarchical set by voting differently. The second extension is called *coalitional pivotal sets*. A coalitional pivotal set is composed of groups of voters who can change the winner by voting differently in collaboration. Based on the coalitional pivotal sets, we can use Shapley-Shubik power index as a measurement of voting power in the induced coalitional game.[2]

Our technical contributions. The main technical contribution of this paper is the following characterization of the hierarchical pivotal sets. We prove that for any voting rule that satisfies *anonymity* and *unanimity* and any profile, the voters in the hierarchical pivotal sets are not redundant (a voter is redundant if he/she is never pivotal in any profile). And conversely, any non-redundant voter must be in the level-k pivotal set for some $k \leq n+1$, where n is the number of voters. Therefore, in terms of hierarchical pivotal sets, for any anonymous voting rule, in any profile, any voter has some voting power to (directly or indirectly) change the winner. This provides a new perspective towards understanding the paradox of not voting. However, we also show that there exists a non-anonymous voting rule, where for any given profile, not all non-redundant voters are in the union of all hierarchical pivotal sets.

We also investigate algorithmic aspects of computing the hierarchical pivotal sets and coalitional pivotal sets. We study the relations among computing the two types of pivotal sets, the unweighted coalitional manipulation problem, and computing the margin of victory. These connections together with previous work give us characterizations of the computational complexity of checking whether the two types pivotal sets are empty for some common voting rules. We also propose a dynamic programming algorithm to compute the level-k pivotal sets for anonymous voting rules. The runtime of the algorithm is polynomial if the number of alternatives is bounded above by a constant.

Preliminaries

Let $\mathcal{C}$ be a finite set of *alternatives* (or *candidates*). A *vote* V is a linear order over $\mathcal{C}$, i.e., a transitive, antisymmetric, and total relation over $\mathcal{C}$. The set of all linear orders over $\mathcal{C}$ is denoted by $L(\mathcal{C})$. An n-voter profile P over $\mathcal{C}$ is a collection of n linear orders over $\mathcal{C}$, that is, $P = (V_1, \ldots, V_n)$, where for every $j \leq n$, $V_j \in L(\mathcal{C})$. In this paper, we let m denote the number of alternatives and let n denote the number of voters (agents) in a profile. Let $N = \{1, \ldots, n\}$. For any subset $S \subseteq N$, we let P_S denote the sub-profile of P that consists of the votes of the voters in S; let $P_{-S} = P_{N\setminus S}$. When $S = \{i\}$, we write P_{-i} instead of $P_{-\{i\}}$. The set of all n-profiles over $L(\mathcal{C})$ is denoted by $\mathrm{F}_n(\mathcal{C})$. In this paper, a *(voting) rule* r maps any n-profile to a single winning alternative, called the *winner*. Some commonly used voting rules are listed below.

- *Positional scoring rules.* Given a *scoring vector* $\vec{v} = (v_1, \ldots, v_m)$ of m integers, for any vote $V \in L(\mathcal{C})$ and any $c \in \mathcal{C}$, let $s_{\vec{v}}(V, c) = v_i$, where i is the rank of c in V. For any profile $P = (V_1, \ldots, V_n)$, let $s_{\vec{v}}(P, c) = \sum_{j=1}^{n} s_{\vec{v}}(V_j, c)$. The rule will select an alternative $c \in \mathcal{C}$ so that $s_{\vec{v}}(P, c)$ is maximized. Some examples of positional scoring rules are *plurality*, for which the scoring vector is $(1, 0, \ldots, 0)$, and *veto*, for which the scoring vector is $(1, \ldots, 1, 0)$. Plurality is also called *majority* when there are only two alternatives.
- *Single transferable vote (STV).* The election has m rounds. In each round, the alternative that gets the minimal plurality score drops out, and is removed from all of the votes. The last-remaining alternative is the winner.
- *Ranked pairs.* This rule first creates an entire ranking of all the alternatives. Let $D_P(c_i, c_j)$ denote the number of votes where $c_i \succ c_j$ minus the number of votes where $c_j \succ c_i$ in the profile P. In each step, we consider a pair of alternatives c_i, c_j that we have not previously considered, which has the highest $D_P(c_i, c_j)$ among the remaining pairs. We then fix the order $c_i \succ c_j$, unless it violates transitivity. We continue until all pairs of alternatives have been considered. The alternative at the top of the ranking wins.

A voting rule r satisfies *anonymity*, if the winner under r does not depend on the name of the voters. That is, for any permutation M over N and any profile $P = (V_1, \ldots, V_n)$, we have $r(P) = r(M(P)) = r(V_{M(1)}, \ldots, V_{M(n)})$. r satisfies *unanimity*, if for any profile P in which all voters rank the same alternative c in their top positions, $r(P) = c$.

Two new extensions of pivotal sets

In this section, we introduce two extensions of pivotal sets and discuss their relationship.

Hierarchical pivotal sets

Hierarchical pivotal sets are defined recursively. Given a voting rule r and a profile P, the level-1 pivotal set $\mathrm{PS}_r^1(P) \subseteq N$ is defined to be the set of all pivotal voters. That is, $j \in \mathrm{PS}_r^1(P)$ if and only if there exists a vote V_j' such that $r(P_{-j}, V_j') \neq r(P)$.

The level-2 pivotal sets to be composed of all voters who can change the level-1 pivotal set by voting differently. More generally, for any natural number k, we define the level-k pivotal set $\mathrm{PS}_r^k(P) \subseteq N$ recursively as follows.

Definition 1 *For any voting rule r, any $k \in \mathbb{N}$, and any profile P, we define the level-k pivotal set $PS_r^k(P) \subseteq N$ recursively as follows.*

- *$j \in PS_r^1(P)$ if and only if there exists a vote V_j' such that $r(P) \neq r(P_{-j}, V_j')$.*

[2]The concept of coalitional pivotal sets is not new, for example, it is implicitly considered in the coalitional manipulation problems, and it is closely related to the *margin of victory*. However, to the best of our knowledge, our paper is the first time it is used to define voting power.

• *$j \in PS_r^k(P)$ if and only if there exists a vote V_j' such that $PS_r^{k-1}(P) \neq PS_r^{k-1}(P_{-j}, V_j')$. That is, voter j can change the level-$(k-1)$ pivotal set by voting differently.*

Here k is called the *hierarchical level.* Level-k pivotal sets capture voters' indirect power in the current profile P. The higher the hierarchical level is, the more indirectly the voters in it can influence the outcome for P. We note that the level-k pivotal sets for different profiles can be different.

Example 1 Suppose there are two alternatives $\{a, b\}$, 5 voters, and we use the majority rule. Table 1 shows the level-k pivotal sets for all profiles, for $k = 1, 2, 3, 4$. Because the majority rule is anonymous, as we will show later in the paper (Lemma 1), the level-k pivotal set can be represented by a set of votes instead of a set of voters. A pivotal set is denoted by "b" if it is exactly the set of all voters whose votes are $b \succ a$; similarly for "a"; "all" denotes the set of all voters. For example, if two voters vote for $a \succ b$ and three voters vote for $b \succ a$, then the level-3 pivotal set consists of exactly the two voters whose votes are $a \succ b$.

# of $a \succ b$	Pivotal sets				
	1	2	3	4	...
0	$\emptyset$	$\emptyset$	all	$\emptyset$	...
1	$\emptyset$	b	all	b	...
2	b	all	a	all	...
3	a	all	b	all	...
4	$\emptyset$	a	all	a	...
5	$\emptyset$	$\emptyset$	all	$\emptyset$	...

Table 1: The pivotal sets under majority.

Coalitional pivotal sets

When defining hierarchical pivotal sets, we are concerned with the voting power for a single voter to (indirectly) change the winner. It is natural to consider the voting power for a coalition of voters to change the winner by voting collaboratively. We first define the set of pivotal coalitions.

Given a profile P, a subset $S \subset N$ is a *pivotal coalition*, if there exists a profile P_S' for the voters in S such that $r(P) \neq r(P_{-S}, P_S')$. We define the indicator function v_r^P as follows. For any coalition $S \subseteq N$, if S is a pivotal coalition, then $v_r^P(S) = 1$; otherwise $v_r^P(S) = 0$. For any voting rule r and any profile P, let $\mathrm{CPS}_r(P)$ denote the set of all pivotal coalitions, that is, $\mathrm{CPS}_r(P) = \{S \subseteq N : v_r^P(S) = 1\}$. $\mathrm{CPS}_r(P)$ is called a *coalitional pivotal set.*

Obviously, if a set of voters S can change the winner, then any superset of S can also change the winner. Therefore, for any r and any profile P, $\mathrm{CPS}_r(P)$ is *upward-closed*, that is, for any $S \in \mathrm{CPS}_r(P)$ and any S' such that $S \subseteq S'$, we have $S' \in \mathrm{CPS}_r(P)$.

Example 2 *There are three alternatives $\{a, b, c\}$. Let P=($a \succ b \succ c$, $a \succ c \succ b$, $c \succ a \succ b$). We have $CPS_{\mathrm{Plu}}(P) = \{\{1\}, \{2\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$ and $CPS_{\mathrm{Veto}}(P) = \{\{1\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.*

Given a voting rule r and a profile P, the coalitional pivotal set $\mathrm{CPS}_r(P)$ naturally induces a coalitional game, where a group of agents S is winning if and only if $S \in \mathrm{CPS}_r(P)$. Hence, we can measure a voter's voting power by the Shapley-Shubik power index (Shapley and Shubik 1954), defined as follows. To the best of our knowledge, our paper is the first time that Shapley-Shubik power index is considered in the context of preference aggregation by voting rules. Let $w_r : \mathrm{F}_n(\mathcal{C}) \times N \to \mathbb{R}_{\geq 0}$ be a mapping such that for any profile $P \in \mathrm{F}_n(\mathcal{C})$ and any $j \leq n$, we have:

$$w_r(P, j) = \sum_{S \subseteq N \setminus \{j\}} \frac{|S|!(n - |S| - 1)!}{n!}(v_r^P(S \cup \{j\}) - v_r^P(S))$$

Relationships between the two types of pivotal sets

The next theorem states that the smallest k such that the level-k pivotal set is non-empty equals to the size of the smallest coalitional pivotal set for P.

Theorem 1 *For any voting rule r and any profile P,*
$$\min\{k : PS_r^k(P) \neq \emptyset\} = \min_{S \in CPS_r(P)}\{|S|\}$$

Proof: Let $k^* = \arg\min_k\{\mathrm{PS}_r^k(P)\}$ and $k' = \min_{S \in \mathrm{CPS}_r(P)}\{|S|\}$. We first prove that $k^* \leq k'$. Suppose for the sake of contradiction that $k^* > k'$. Without loss of generality, $S = \{1, \ldots, k'\}$, and let $P_S' = (V_1', \ldots, V_{k'}')$ be the votes such that $r(P) \neq r(P_{-S}, P_S')$. For any $k \leq k'$, let $P_k = (V_1', \ldots, V_k', V_{k+1}, \ldots, V_n)$, that is, P_k is obtained from P by replacing the first k votes by $V_1', \ldots, V_k'$, respectively. Because $k' < k^*$, for any $k \leq k'$, $\mathrm{PS}_r^k(P) = \emptyset$. Therefore, for any $k \leq k' - 1$, changing the vote of voter 1 from V_1 to V_1' does not change the level-k pivotal set. That is, for any $k \leq k' - 1$, $\mathrm{PS}_r^k(P_1) = \emptyset$. Similarly, it is easy to see that for any $i \leq k' - 1$, for any $k \leq k' - i$, $\mathrm{PS}_r^k(P_i) = \emptyset$. Specifically, $\mathrm{PS}_r^1(P_{k'-1}) = \emptyset$. It follows from $\mathrm{PS}_r^1(P) = \emptyset$ and for any $i \leq k' - 1$, $\mathrm{PS}_r^1(P_i) = \emptyset$, that $r(P) = r(P_1) = r(P_2) = \ldots = r(P_{k'})$. This contradicts the assumption that $r(P) \neq r(P_{k'})$. Consequently, $k^* \leq k'$.

Next, we prove that $k' \leq k^*$. It suffices to prove that for any $k \leq k' - 1$, $\mathrm{PS}_r^k(P) = \emptyset$. We have following stronger claim, whose proof is omitted due to the space constraint.

Claim 1 *For any $2 \leq q \leq k'$, any P' that differs from P on no more than $k' - q$ votes, and any $k \leq q - 1$, $PS_r^k(P') = \emptyset$.*

Proof of Claim 1: We prove the claim by induction on q. Because any coalition whose size is strictly smaller than k' cannot change the winner, for any P_2 that differs from P on at most $k' - 2$ votes and any $\bar{P}_2$ that differs from P_2 on one vote, we must have that $r(P_2) = r(\bar{P}_2)$. This means that $\mathrm{PS}_r^1(P_2) = \emptyset$. Therefore, the claim holds for $q = 2$.

Now suppose the claim holds for $q = q'$. Let $P_{q'+1}$ be a profile that differs from P on at most $k' - q' - 1$ votes, and let $\bar{P}_{q'+1}$ be a profile that differs from $P_{q'+1}$ on one vote. It follows that $\bar{P}_{q'+1}$ differs from P on at most $k - q'$ votes. By the induction hypothesis, for any $k \leq q' - 1$, $\mathrm{PS}_r^k(P_{q'+1}) = \mathrm{PS}_r^k(\bar{P}_{q'+1}) = \emptyset$. Specifically, $\mathrm{PS}_r^{q'-1}(P_{q'+1}) = \mathrm{PS}_r^{q'-1}(\bar{P}_{q'+1}) = \emptyset$. Therefore, $\mathrm{PS}_r^{q'}(P_{q'+1}) = \emptyset$, which means that the claim holds for $q = q' + 1$. It follows that the claim holds for all $q \leq k'$. □

Let $q = k'$ in Claim 1, we have that $\mathrm{PS}_r^{k'-1}(P) = \emptyset$, which means that $k^* \geq k'$. Therefore, $k^* = k'$. □

Hierarchical pivotal sets for anonymous voting rules

In this section, we characterize the hierarchical pivotal sets for anonymous voting rules. It is easy to see that if a voter is not pivotal in *any* profile, then for any k and any profile P, she is not in the level-k pivotal set. Such a voter is said to be *redundant*.

Definition 2 *Given a voting rule r, a voter j is* redundant, *if for any profile P and any vote V_j', $r(P) = r(P_{-j}, V_j')$.*

If a voter is redundant, then effectively her vote can be completely ignored. Therefore, for any profile, none of the voters in the union of its hierarchical pivotal sets (as $k \to \infty$) is redundant. That is, the union of the hierarchical pivotal sets for any profile is a *sound* characterization of the non-redundant voters. We ask the following two natural questions. The first question asks whether or not the union of the hierarchical pivotal sets for a given profile P is a *complete* characterization of the non-redundant voters. A positive answer to this question means that any non-redundant voter has some voting power in terms of being level-k pivotal for some k.

Question 1 *Given a voting rule r, is it true that for any non-redundant voter j and any profile P, there exists $k \in \mathbb{N}$ such that j is in the level-k pivotal set for P?*

The second question concerns the asymptotic property of level-k pivotal sets when k goes to infinity. Given a profile P, we are asked whether the level-k pivotal sets for P will converge (to the empty set), when k goes to infinity.

Question 2 *Given a voting rule r, does there exist $K \in \mathbb{N}$ such that for any $k \geq K$, the level k-pivotal set is $\emptyset$?*

In this section, we give an affirmative answer to Question 1 for any voting rule that satisfies anonymity and unanimity, and a negative answer to Question 2 for the majority rule. We first prove a lemma, which states that for any anonymous voting rule r, if a voter j is in the level-k pivotal set for a profile P, then other voters who cast the same vote as j's vote are also in the level-k pivotal set for P. This lemma will be frequently used in this paper.

Lemma 1 *For any anonymous voting rule r, any profile P, any $k \in \mathbb{N}$, and any pair of voters i, j with $V_i = V_j$, $i \in PS_r^k(P)$ if and only if $j \in PS_r^k(P)$.*

Proof: We prove a more general claim, which states that for any anonymous voting rule r and any profile P, if we switch the votes of voter i and voter j (with $i \neq j$) and let P' denote the resulting profile, then for any k, in $\mathrm{PS}_r^k(P')$, i should be replaced by j, and j should be replaced by i. Let $M_{i,j}$ denote the permutation over $\{1, \ldots, n\}$ that only exchanges i and j.

Claim 2 *For any anonymous voting rule r, any profile P, and any $k \in \mathbb{N}$, we have $PS_r^k(P') = M_{i,j}(PS_r^k(P))$, where P' is the profile obtained from P by exchanging the votes of voter i and j.*

Proof of Claim 2: Intuitively, the proposition directly follows from the anonymity of r. Formally, we prove the claim by induction on k. We have $P' = M_{i,j}(P) = (V_{M_{i,j}(1)}, V_{M_{i,j}(2)}, \ldots, V_{M_{i,j}(n)})$.

When $k = 1$, if $i \in \mathrm{PS}_r^1(P)$, then there exists a vote W such that $r(P) \neq r(P_{-i}, W)$. Because r is anonymous, we have $r(M_{i,j}(P)) = r(P) \neq r(P_{-i}, W) = r(M_{i,j}(P_{-i}, W)) = r((M_{i,j}(P))_{-j}, W)$. Therefore, $j \in \mathrm{PS}_r^1(P')$. Similarly we can prove that if $j \in \mathrm{PS}_r^1(P)$, then $i \in \mathrm{PS}_r^1(P')$. For any l such that $l \neq i$ and $l \neq j$, and any vote W, we have $r(P_{-l}, W) = r((M_{i,j}(P))_{-l}, W)$. Therefore, $r(P) \neq r(P_{-l}, W)$ if and only if $r(M_{i,j}(P)) \neq r((M_{i,j}(P))_{-l}, W)$, which means that $l \in \mathrm{PS}_r^1(P)$ if and only if $l \in \mathrm{PS}_r^1(M_{i,j}(P))$. It follows that $\mathrm{PS}_r^1(M_{i,j}(P)) = M_{i,j}(\mathrm{PS}_r^1(P))$.

Suppose the claim holds for all $k \leq k'$, we now prove that the claim also holds for $k = k' + 1$. If $i \in \mathrm{PS}_r^{k'+1}(P)$, then there exists a vote W such that $\mathrm{PS}_r^{k'}(P) \neq \mathrm{PS}_r^{k'}(P_{-i}, W)$. By the induction hypothesis, we have the following calculation.

$$\begin{aligned}
&\mathrm{PS}_r^{k'}(M_{i,j}(P)) = M_{i,j}(\mathrm{PS}_r^{k'}(P)) \\
\neq &M_{i,j}(\mathrm{PS}_r^{k'}(P_{-i}, W)) = \mathrm{PS}_r^{k'}(M_{i,j}(P_{-i}, W)) \\
= &\mathrm{PS}_r^{k'}((M_{i,j}(P))_{-j}, W)
\end{aligned}$$

Therefore $j \in \mathrm{PS}_r^{k'+1}(P')$. Similarly we can prove that if $j \in \mathrm{PS}_r^{k'+1}(P)$, then $i \in \mathrm{PS}_r^{k'+1}(P')$. For any l such that $l \neq i$ and $l \neq j$, and any vote W, by the induction hypothesis we have $M_{i,j}(\mathrm{PS}_r^{k'}(P_{-l}, W)) = \mathrm{PS}_r^{k'}(M_{i,j}(P_{-l}, W)) = \mathrm{PS}_r^{k'}(M_{i,j}(P))_{-l}, W)$. It follows that $\mathrm{PS}_r^{k'}(P) \neq \mathrm{PS}_r^{k'}(P_{-l}, W)$ if and only if $\mathrm{PS}_r^{k'}(M_{i,j}(P)) \neq \mathrm{PS}_r^{k'}(M_{i,j}(P)_{-l}, W)$, which means that $l \in \mathrm{PS}_r^{k'+1}(P)$ if and only if $l \in \mathrm{PS}_r^{k'+1}(M_{i,j}(P))$. Hence, we have $\mathrm{PS}_r^{k'+1}(M_{i,j}(P)) = M_{i,j}(\mathrm{PS}_r^{k'+1}(P))$, which means that the lemma holds for $k = k'+1$. Therefore, the claim holds for any $k \in \mathbb{N}$. □

Because $V_i = V_j$, $M_{i,j}(P) = P$, which means that $\text{PS}_r^k(P) = M_{i,j}(\text{PS}_r^k(P))$. The lemma follows directly from Claim 2. □

It follows from Lemma 1 that for any anonymous voting rule r and any profile P, a voter's membership in the level-k pivotal set can be characterized by her vote. Therefore, for any anonymous voting rule r and any profile, the level-k pivotal set can be represented by the set of all votes that are cast by some level-k pivotal voters. The next theorem gives an affirmative answer to Question 1 for any voting rule that satisfies anonymity and unanimity.

Theorem 2 *Let r be a voting rule that satisfies anonymity and unanimity. For any n-profile P and any voter j, there exists $k \leq \min_{S \in CPS_r(P)}\{|S|\} + 1 \leq n+1$ such that $j \in PS_r^k(P)$.*

Proof: Let $K = \min_{S \in \text{CPS}_r(P)}\{|S|\}$. For the sake of contradiction, without loss of generality for any $k \leq K+1$, $1 \notin \text{PS}_r^k(P)$. By Theorem 1, there exists $k^* \leq K$ such that $\text{PS}_r^{k^*}(P) \neq \emptyset$. Let $j^* \in \text{PS}_r^{k^*}(P)$ and W be the vote of voter j^*. Let $P' = (P_{-1}, W)$, that is, P' is the profile obtained from P by letting voter 1 vote for W. Because $1 \notin \text{PS}_r^{k^*}(P)$ and $1 \notin \text{PS}_r^{k^*+1}(P)$, we have that $1 \notin \text{PS}_r^{k^*}(P')$. It follows from Lemma 1 that for any voter j whose vote is W in P', $j \notin \text{PS}_r^{k^*}(P')$. Specifically, $j^* \notin \text{PS}_r^{k^*}(P')$, which means that $\text{PS}_r^{k^*}(P') \neq \text{PS}_r^{k^*}(P)$. Therefore, $1 \in \text{PS}_r^{k^*+1}(P)$. This contracts the assumption that $1 \notin \text{PS}_r^{k^*+1}(P)$. □

Theorem 2 is quite positive. It states that for any voting rule that satisfies anonymity and unanimity, any voter has some voting power in terms of being level-k pivotal for some k.

For Question 2, suppose the level-k pivotal set converges as k goes to infinity, we first prove that it must converge to $\emptyset$.

Proposition 1 *For any anonymous voting rule r, if there exists k such that for every n-profile P, $PS_r^k(P) = PS_r^{k+1}(P)$, then for every n-profile P, $PS_r^k(P) = \emptyset$.*

Proof: For the sake of contradiction, suppose for every n-profile $\bar{P}$, $\text{PS}_r^k(\bar{P}) = \text{PS}_r^{k+1}(\bar{P})$ and there exists an n-profile P such that $\text{PS}_r^k(P) \neq \emptyset$. We prove the proposition for the following two cases.

Case 1: $\text{PS}_r^k(P) \neq \{1, \ldots, n\}$. We arbitrarily choose j' and j^* such that $j' \notin \text{PS}_r^k(P)$ and $j^* \in \text{PS}_r^k(P)$. Because $j' \notin \text{PS}_r^{k+1}(P) = \text{PS}_r^k(P)$, for every vote W, $\text{PS}_r^k(P_{-j'}, W) = \text{PS}_r^k(P)$. Specifically, $\text{PS}_r^k(P_{-j'}, V_{j^*}) = \text{PS}_r^k(P)$. That is, if voter j' change her vote to the vote of voter j^*, then the level k pivotal set does not change. However, since $j^* \in \text{PS}_r^k(P) = \text{PS}_r^k(P_{-j'}, V_{j^*})$ and in $(P_{-j'}, V_{j^*})$, the vote of voter j' is the same as the vote of voter j^*. By Lemma 1, $j' \in \text{PS}_r^k(P_{-j'}, V_{j^*}) = \text{PS}_r^k(P)$, which contradicts the assumption that $j' \notin \text{PS}_r^k(P)$.

Case 2: For every profile P, either $\text{PS}_r^k(P) = \text{PS}_r^{k+1}(P) = \emptyset$ or $\text{PS}_r^k(P) = \text{PS}_r^{k+1}(P) = \{1, \ldots, n\}$, and there exists at least one profile P' such that $\text{PS}_r^k(P') = \{1, \ldots, n\}$. If for every profile P, $\text{PS}_r^k(P) = \{1, \ldots, n\}$, then $\text{PS}_r^{k+1}(P) = \emptyset = \text{PS}_r^k(P)$, which is a contradiction. Therefore, there exists a profile P^* such that $\text{PS}_r^k(P^*) = \emptyset$. Let $P' = (V'_1, \ldots, V'_n)$ and $P^* = (V^*_1, \ldots, V^*_n)$. For every $0 \leq j \leq n$, let $P_j = (V^*_1, \ldots, V^*_j, V'_{j+1}, \ldots, V'_n)$. That is, P_j is obtained from P' by letting the first j voters switch to their votes in P^*. We have $P_0 = P'$ and $P_n = P^*$. Therefore, there exists $j \leq n$ such that $\text{PS}_r^k(P_{j-1}) = \text{PS}_r^{k+1}(P_{j-1}) = \{1, \ldots, n\}$ and $\text{PS}_r^k(P_j) = \text{PS}_r^{k+1}(P_j) = \emptyset$. However, since $\text{PS}_r^{k+1}(P_j) = \emptyset$, changing voter j's vote should not change PS_r^k. Therefore, we have $\emptyset = \text{PS}_r^k(P_j) = \text{PS}_r^k((P_j)_{-j}, V'_j) = \text{PS}_r^k(P_{j-1}) = \{1, \ldots, n\}$, which is a contradiction.

Therefore, for any profile P, $\text{PS}_r^k(P) = \emptyset$. □

However, Proposition 1 does not guarantee the existence of k such that $\text{PS}_r^k(P) = \text{PS}_r^{k+1}(P)$. In fact, the next proposition shows that such a k might not exist for the majority rule, which satisfies anonymity and unanimity. Therefore, the answer to Question 2 is negative.

Proposition 2 *Let there be two alternatives $\{a, b\}$, 5 voters, and we use the majority rule. There does not exist $k \in \mathbb{N}$ such that for any profile P, the level-k pivotal set for P is $\emptyset$.*

Proof: From Table 1 in Example 1, it is easy to see that for any profile, its level-2 and level-4 pivotal sets are identical and are different from level-3 pivotal sets. Therefore, for any profile, none of the level-k pivotal sets converges as k goes to infinity. □

Hierarchical pivotal sets for non-anonymous voting rules

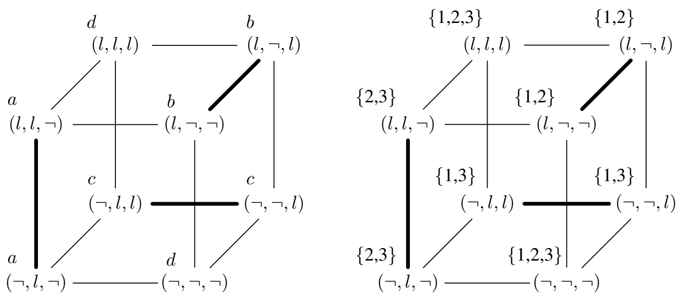
In this section, we focus on non-anonymous voting rules. Surprisingly, for some voting rules that do not satisfy anonymity, the answer to Question 1 is negative.

Proposition 3 *Let $m = 4$ and $n = 3$. There exists a non-anonymous voting rule r that satisfies the following conditions.*

- *No voter is redundant.*
- *For any $k \in \mathbb{N}$ and any profile P such that $|P| = 3$, the level-k pivotal set for P is non-empty.*
- *For any voter j, there exists a profile P such that $|P| = 3$ and for any $k \in \mathbb{N}$, j is not in the level-k pivotal set for P.*

Proof: Let the four alternatives be $\{a, b, c, d\}$. Let $l = [a \succ b \succ c \succ d]$. We define a voting rule r as follows. $r(l, l, \neg) = r(\neg, l, \neg) = a$, $r(l, \neg, l) = r(l, \neg, \neg) = b$, $r(\neg, l, l) = r(\neg, \neg, l) = c$, $r(l, l, l) = r(\neg, \neg, \neg) = d$.

Here "$\neg$" means any linear order that is different from l. For example, $r(\neg, l, l) = c$ means that for any 3-profile where voter 1's voter is not l, and the votes of voter 2 voter and 3 are both l, the winner is c. The voting rule is illustrated in Figure 1(a), where each vertex represents a set of 3-profiles and the alternative associated with it is the winner for these profiles. An edge between two vertices A and B in the graph means that for any profile P in A, there exists a profile P' in B such that P' can be obtained from P by changing exactly one vote. An edge is bold if the winners for its two endpoints are the same. We have the following claim.



(a) The voting rule r. (b) The level-k pivotal set for any k.

Figure 1: The voting rule r and the hierarchical pivotal sets.

Claim 3 *For any $k \in \mathbb{N}$ and any profile P, $PS_r^k(P) = PS_r^{k+1}(P)$, and is illustrated in Figure 1 (b).*

Proof: The claim is proved by induction on k. The case where $k = 1$ is easy to verify. Due to the space constraint, we only show an example of computing the level-$(k + 1)$ pivotal set for profiles in $(\neg, l, \neg)$ given that the level-k pivotal sets are shown in Figure 1 (b). Let P be a profile in $(\neg, l, \neg)$. That is, $P = (l', l, l^*)$ with $l' \neq l$ and $l^* \neq l$. We first show that $1 \notin \text{PS}_r^{k+1}(P)$. For any profile P' obtained from P by letting voter 1 vote differently, either $P' \in (\neg, l, \neg)$ or $P' \in (l, l, \neg)$. By the assumption that the level-k pivotal sets are illustrated in Figure 1 (b), we have that $\text{PS}_r^k(\neg, l, \neg) = \text{PS}_r^k(l, l, \neg) = \{2, 3\}$. Hence, voter 1 cannot change the level-k pivotal set for P by voting differently, which means that $1 \notin \text{PS}_r^{k+1}(P)$. We note that $\text{PS}_r^k(l', l', l^*) = \{1, 2, 3\} \neq \{2, 3\}$ (the profile belongs to the vertex $(\neg, \neg, \neg)$), and $\text{PS}_r^k(l', l, l) = \{1, 3\} \neq \{2, 3\}$ (the profile belongs to the vertex $(\neg, l, l)$). Therefore, $\text{PS}_r^{k+1}(P) = \{2, 3\}$. Similarly, it is easy to check that for each of the eight vertex, the level-$(k+1)$ pivotal set is the same as depicted in Figure 1 (b). □

It follows from Claim 3 that r satisfies all the properties in the description of the proposition. □

Computing hierarchical pivotal sets and coalitional pivotal sets

In this section, we investigate the computational complexity of computing level-k pivotal sets. We first relate the problem of computing level-1 pivotal sets to the *unweighted coalitional manipulation (UCM)* problems with a single manipulator. An instance of UCM is a tuple (r, P^{NM}, c, M), where r is a voting rule, P^{NM} is the non-manipulators' profile, c is the manipulators' preferable alternative, and M is the set of manipulators. We are asked whether there exists a profile P^M for the manipulators such that $r(P^{NM} \cup P^M) = c$. Let UCM_1 denote the UCM problems with a single manipulator, that is, $|M| = 1$.

Proposition 4 *For any voting rule r, if UCM_1 is in P, then deciding whether a given voter is in $PS_r^1(P)$ is also in P.*

Following the results of computing UCM_1 for common voting rules (Bartholdi, Tovey, and Trick 1989; Bartholdi and Orlin 1991; Conitzer, Sandholm, and Lang 2007; Faliszewski, Hemaspaandra, and Schnoor 2008; Zuckerman, Procaccia, and Rosenschein 2009; Xia et al. 2009), we immediately obtain the following corollary.

Corollary 1 *For any $r \in$ {Copeland, Veto, Plurality with runoff, Cup, Maximin, Bucklin, Borda} and any profile P, there exists a polynomial-time algorithm that decides whether a given voter is in $PS_r^1(P)$.*[3]

Computing the hierarchical pivotal sets and coalitional pivotal sets is also closely related to computing the *margin of victory* of elections.

Definition 3 *Given a voting rule r and a profile P, The margin of victory (MoV) of P w.r.t. r, is the smallest number of voters who can change the winner by voting differently, while the other voters keep their votes unchanged.*

The margin of victory is an important measurement for closeness of the election. It also plays an important role in implementing efficient post-election audits, which is an important technique to detect errors or frauds in elections. Therefore, it is important to know the margin of victory for some common voting rules. The computational problem is defined as follows. Given a voting rule r, a profile P, and a natural number k, we are asked whether the margin of victory of P is no more than k. This problem is denoted by MoV_k. The computational complexity as well as practical algorithms for MoV_k has been investigated recently (Magrino et al. 2011; Cary 2011; Rivest, Shen, and Xia 2011).

We note that MoV_k is equivalent to asking whether $\text{CPS}_r(P)$ contains a set S such that $|S| \leq k$. By Theorem 1, these two problems are also equivalent to asking whether there exists $k' \leq k$ such that $\text{PS}_r^{k'}(P) \neq \emptyset$. Therefore, the following corollary follows directly from the results in (Rivest, Shen, and Xia 2011).

Proposition 5 *Given a voting rule r, a profile P, and a natural number k, deciding whether $CPS_r(P)$ contains*

[3]The definition of these voting rules can be found in e.g. (Xia et al. 2009).

a set whose cardinality is no more than k and deciding whether $PS_r^{k'}(P)$ is non-empty for some $k' \leq k$ is

- *in* P *for all positional scoring rules and plurality with runoff;*
- NP*-complete for Copeland and maximin;*
- NP*-complete for STV and ranked pairs even for $k = 1$.*

For any anonymous voting rule, when m is bounded above by a constant, we have a dynamic-programming algorithm that computes the level-k pivotal set. The algorithm is based on the following two key observations. First, when the number of alternatives is bounded above by a constant, the number of essentially different profiles is polynomial. Second, by Lemma 1, a level-k pivotal set can be represented succinctly by a set of votes (instead of voters).

Let $D_{m,n}$ denote the set of all $m!$-dimensional integer vectors whose components sum up to n. That is, $D_{m,n} = \{(d_1, \ldots, d_{m!}) \in \mathbb{N}^{m!} : \sum_{i=1}^{m!} d_i = n\}$. In the social choice literature, a vector in $D_{m,n}$ is called a *voting situation* (Berg and Lepelley 1994), which uniquely represents an n-profile for anonymous voting rules (the ith component represents the number of voters who cast l_i in the profile). For any $i \leq m!$, let $\vec{e}_i \in \mathbb{N}^{m!}$ denote the vector whose ith component is 1, and all the other components are 0. For any anonymous voting rule r and any $k \in \mathbb{N}$, we let $\mathrm{VPS}_r^k : D_{m,n} \to 2^{L(\mathcal{C})}$ be the mapping such that for any profile $\vec{d} \in D_{m,n}$, $\mathrm{VPS}_r^k(\vec{d})$ is the set of all level-k pivotal votes in $\vec{d}$. That is, any voter whose vote is in $\mathrm{VPS}_r^k(\vec{d})$ is level-k pivotal for $\vec{d}$. For any profile $\vec{d} \in D_{m,n}$ and any $i \leq m!$, if $d_i > 0$, then we let $f_i(\vec{d}) = \{\vec{d} - \vec{e}_i + \vec{e}_{i'} : i' \leq m!\}$; if $d_i = 0$, then we let $f_i(\vec{d}) = \emptyset$. Algorithm 1 computes VPS_r^k as follows.

Algorithm 1: CompPivotalSet

Input: r, k.
Output: VPS_r^k.
For any $\vec{d'} \in D_{m,n}$, let $\mathrm{VPS}_r^0(\vec{d'}) = \{r(\vec{d'})\}$.
for $j = 1$ *to* k **do**
  for *each* $\vec{d'} \in D_{m,n}$ *and each* $i \leq m!$ **do**
    Compute $f_i(\vec{d'}) = \{\vec{d'} - \vec{e}_i + \vec{e}_{i'} : i' \leq m!\}$.
    for *each* $\vec{h} \in f_i(\vec{d'})$ **do**
      if $f_i(\vec{d'}) \neq \emptyset$ *and* $VPS_r^{j-1}(\vec{d'}) \neq VPS_r^{j-1}(\vec{h})$ **then**
        $\mathrm{VPS}_r^j(\vec{d'}) \leftarrow l_i$.
      end
    end
  end
end
return VPS_r^k.

We immediately have the following proposition whose proof is straightforward and is thus omitted.

Proposition 6 *Suppose m is bounded above by a constant. For any anonymous voting rule r and any $k \in \mathbb{N}$, Algorithm 1 computes the level-k pivotal sets in polynomial time (in k and n).*

For plurality and veto, we can further reduce the complexity by using an m-dimensional vector to represent a profile, where the ith component represents how many times the ith alternative is ranked in the top/bottom position. Let $\hat{D}_{m,n}$ denote the set of all such vectors. We have $|\hat{D}_{m,n}| \leq m^n$. For any profile $\vec{d} \in \hat{D}_{m,n}$ and any $i \leq m$, if $d_i > 0$, then we let $\hat{f}_i(\vec{d}) = \{\vec{d} - \vec{e}_i + \vec{e}_{i'} : i' \leq m\}$; if $d_i = 0$, then we let $\hat{f}_i(\vec{d}) = \emptyset$. We can then tweak Algorithm 1 in the following way: $D_{m,n}$ is replaced by $\hat{D}_{m,n}$ and f_i is replaced by $\hat{f}_i$. Then, it is easy to see that when n is bounded above by a constant, the running time of the algorithm is polynomial in m and k.

Future research

We believe that defining and computing voting power are important. It would be worthwhile studying applications of the two types of voting powers proposed in this paper (especially the Shapley-Shubik power index), for example, they can be used to define random dictatorships or analyze the coalition formation of the manipulators. We can also examine other ways of defining voting power, for example by using the Banzhaf power index (Banzhaf 1965).

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